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ИССЛЕДОВАНИЕ КОГЕРЕНТНЫХ ФОТОННЫХ ПОТОКОВ МЕТОДОМ СОВПАДЕНИЙ

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ОТДЕЛ КОСМИЧЕСКИХ ЛУЧЕЙ ЦЕНТРАЛЬНОГО НАУЧНО-ИССЛЕДОВАТЕЛЬСКОГО ИНСТИТУТА ФИЗИКИ, БУДАПЕШТ

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В этой статье исследуется вопрос, являются фотоны, принадлежащие к двум когерентным световым пучкам, независимыми. Два когерентных световых пучка, осуществленных с помощью полупрозрачного зеркала, падали на фотоэлектронные умножители. Задача заключалась в том, чтобы определить существуют ли истинные совпадения между импульсами фотоумножителей. Для этого сравнивались число совпадений полученных при когерентном освещении с числом случайных совпадений полученных при некогерентном освещении.

Произведенные нами исследования показывают что истинных совпадений нет. Точнее говоря, если принимать во внимание статическую погрешность, то не больше 0,6% фотонов могли бы дать истинные совпадения.

I.

§ 1. Введение

Наше исследование было предпринято с целью установить, независимы ли фотоны в когерентных световых пучках. Согласно квантовой теории, такие фотоны независимы в следующем смысле: если поставить на пути

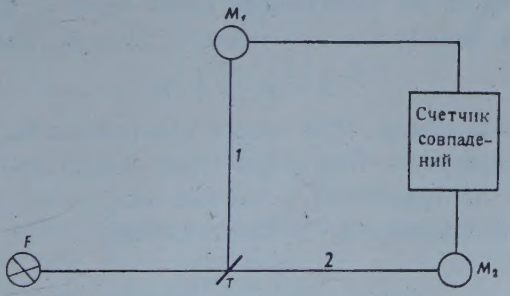

Рис. 1. Схема опыта

светового потока полупрозрачное зеркало, разделяющее его на отраженный и проходящий пучки, а затем улавливать эти пучки счётчиками фотонов, то счётчики будут срабатывать независимо друг от друга, т. е. всякий раз будет срабатывать лишь один из них. Для проверки этого утверждения был поставлен опыт по следующей схеме (рис. 1):

Свет от источника F падает на полупрозрачное зеркало T. Отраженный пучок 1 падает на фотоэлектронный умножитель M_1, а прошедший через зеркало пучок 2 — на фотоумножитель M_2. С помощью соответствующей аппаратуры мы регистрируем т. н. совпадения, т. е. те случаи, когда оба фотоумножителя срабатывают одновременно.

Связь данного опыта с проблемами квантовой механики, — т. е. с вопросом о корпускулярно-волновом характере элементарных частиц подробно освещена Л. Яноши [1, 2]. В тех же статьях автор указывает на важность экспериментального изучения данного вопроса.

§ 2. Случайные совпадения и подбор разрешающего времени аппаратуры, регистрирующей совпадения

Вышеприведенная схема неизбежно считает как истинные, так и случайные совпадения, т. е. совпадения испульсов двух фотоумножителей, вызванных двумя фотонами, приходящими друг за другом через короткий промежуток времени, не превышающий так называемое разрешающее время τ.

Наш эксперимент должен выяснить, не отмечает ли установка при освещении фотоэлектронных умножителей когерентными пучками света больше совпадений, чем ожидаемое число случайных совпадений.

Прежде всего, необходимо определить, возможно ли с уверенностью установить различие между числами систематических и случайных совпадений, несмотря на то, что вероятность срабатывания умножителей мала.

Если обозначить вероятность срабатывания умножителя через p, средний поток падающего на умножитель света через n фотонов в секунду, а число импульсов фона (в темноте) через N_s, то анод умножителя в секунду

$$N = p \cdot n + N_s \tag{1}$$

будет поступать импульсов. При наших измерениях мы охлаждали умножители жидким воздухом. Благодаря этому, число импульсов фона было ничтожно малым по сравнению с числом импульсов, вызванных светом, и им можно было пренебречь. В таком случае

$$N = pn.$$

Как известно, среднее число случайных совпадений при продолжительности измерения t можно определить по формуле

$$K_v = 2 N^2 \tau t, \tag{2}$$

где τ обозначает разрешающее время аппаратуры. (Простоты ради мы предположили, что вероятность срабатывания умножителей одинакова, одинаковы и поступающие на них световые потоки. Ход расчета не меня-

ется, если принять во внимание разные вероятности срабатывания и разницу световых потоков).

Общее число отмеченных совпадений равно числу случайных совпадений, плюс число истинных совпадений. При определении числа истинных совпадений необходимо различать те случаи, когда фотон не вызывает срабатывания обоих умножителей потому, что катод одного из них случайно не испустил электрона (вероятность того, что эмиссия произойдет в обоих умножителях, равна p^2), и те случаи, когда фотон вообще неспособен вызвать срабатывание обоих умножителей, ибо движется исключительно в одном из пучков.

Если предположить, что истинные совпадения способна вызвать ε-ная доля фотонов, т. е. всего εn фотонов, то число таких совпадений составит $\varepsilon p^2 nt$. Общее же число совпадений будет равно:

$$K = K_v + \varepsilon p^2 nt = 2N^2 \tau t + \varepsilon pNt. \tag{3}$$

Если бы истинные совпадения вызывались значительной частью фотонов, т. е. если $\varepsilon \approx 1$, за сравнительно короткое время можно было установить, имеются ли среди совпадений истинные. Расхождение между числами совпадений можно назвать существенным, если

$$K > K_v + 3\Delta K_v, \tag{4}$$

(где через ΔK_v обозначена средняя квадратическая ошибка числа случайных совпадений). Дело в том, что вероятностью появления отклонений, превышающих троекратную среднюю квадратическую ошибку, можно пренебречь. Если в выражение (4) подставить ожидаемую по распределению Пуассона ошибку: $(\Delta K_v)^2 = K_v$, то результат покажет нам, сколько времени должно длиться измерение.

$$\varepsilon pNt > 3\sqrt{2N^2 \tau t}$$

и

$$t > \frac{18}{\varepsilon^2 p^2} \tau. \tag{5}$$

Уже обычный счётчик совпадений легко настроить на разрешающее время $\tau = 2{,}10^{-6}$ сек. Если подставить это — вместе с $\varepsilon = 1$ и $p = 3{,}10^{-3}$, — в формулу (5), мы получим требуемое время: $t > 4$ сек.

Из вышесказанного явствует, что, если бы значительная часть фотонов приводила к истинным совпадениям, это можно было бы установить путем измерения продолжительностью в несколько секунд.

Если только очень малая часть фотонов дает истинные совпадения, т. е. если $\varepsilon \ll 1$, то для установления наличия таких истинных совпадений измерение должно быть довольно продолжительным. Спрашивается, каких результатов можно добиться в случае реально допустимой продолжитель-

ности измерения. Это определяется погрешностью фактора ε. Из формулы (3) следует:

$$\varepsilon = \frac{K - K_v}{pNt} \tag{6}$$

и

$$\Delta\varepsilon = \pm \frac{1}{pNt} \Delta(K - K_v) = \pm \frac{\sqrt{2K_v}}{pNt} = \pm \frac{2}{p} \sqrt{\frac{\tau}{t}}. \tag{7}$$

При выводе выражения (7) мы подставили на основе свойств распределения Пуассона вместо средней квадратичной ошибки корень среднего значения и произвели вследствие $\varepsilon \ll 1$ подстановку $K \approx K_v$. Если подставить в формулу (6) вышеприведенные значения и $t = 5$ часов, мы получим для погрешности ε значение $\Delta\varepsilon = \pm 7{,}10^{-3}$, иными словами: путем пятичасового измерения можно будет установить, вызывает ли истинные совпадения более 2 процентов фотонов.

Отсюда видно, что даже при сравнительно малой вероятности срабатывания фотоэлектронного умножителя можно с помощью обычной установки для регистрации совпадений ($\tau \approx 2{,}10^{-6}$ сек.) определить, превышает ли общее число совпадений число случайных — иными словами, отмечаются ли истинные совпадения.

§ 3. Метод измерения

Чтобы установить, отмечаются ли истинные совпадения, необходимо точно знать число случайных совпадений. Целесообразно поставить опыт

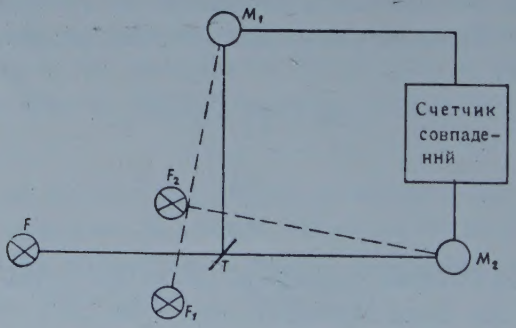

Рис. 2. Установка для счёта истинных и случайных совпадений

так, чтобы число случайных совпадений измерялось непосредственно. Если осветить фотоэлектронные умножители двумя разными источниками света, мы обязательно получим один лишь случайные совпадения. Целесообразно чередовать когерентное и некогерентное освещение и непосредственно сравнивать числа совпадений. (См. рис. 2. Попеременно горят либо источник света F, либо источники света F_1 и F_2.)

II.

§ 4. Оптическая часть устройства

Оптическое оборудование, применявшееся для нашего опыта, изображено на рис. 3.

Из спектра источника света F мы с помощью призменного монохроматора выделили полосу с длиной волны около 5000 Å. Свет проходит через линзу L и отражается зеркалом T на фотоэлектронные умножители M_1 и M^2. Свет от источников F_1 и F_2 попадает соответственно на фотоэлектронные умножители M_1 и M_2, минуя зеркало T. Перед фотоэлектронными умножи-

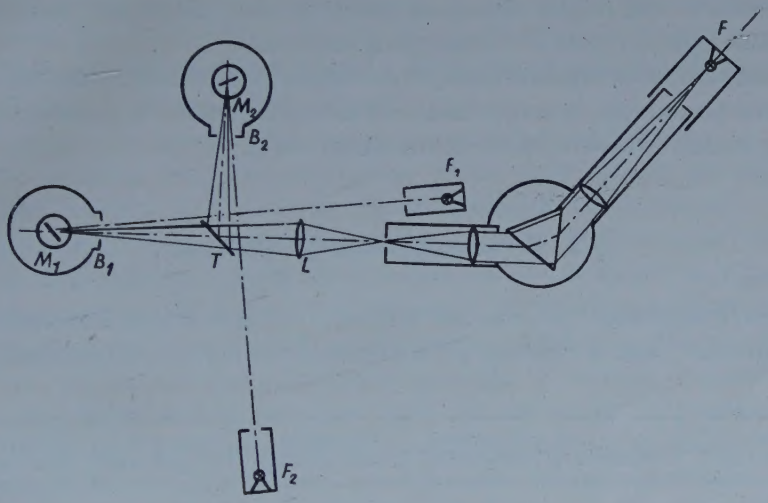

Рис. 3. Оптическая часть устройства

телями в кожухе термостата-охладителя были проделаны отверстия B_1 и B_2 диаметром в 6 миллиметров. В качестве источников света были применены лампы тлеющего разряда типа K_z-12-12. Измерение велось с фотонным потоком в 13 000—130 000 фотонов в секунду. Достижение столь слабого потока не представляло затруднений, ибо : *а*) мы использовали лишь узкую полосу всего спектра, так что интенсивность освещения была и сама по себе невелика, и *б*) перед монохроматором и за ним мы поставили диафрагмы. Тонкая регулировка интенсивности освещения осуществлялась путем изменения тока питания лампочек.

При отрегулировании оптики надо было следить за тем, чтобы с зеркала T весь свет падал на фотокатоды электронных умножителей, ибо в противном случае часть света не могла бы вызывать совпадений. В то же время мы не могли убедиться в правильности наладки установки зрительным путем, вследствие малой интенсивности света и особенностей конструк-

ции охладителя. Поэтому наладка велась в следующей последовательности: источник света F был заменен другим, мощным. Затем мы навели на центры диафрагм B изображения выходного зрачка монохроматора. Эта регулировка велась путем визуального наблюдения. После этого мы снова ставили на место первоначальный источник света и путем передвигания фотоэлектронного умножителя выискивали то положение, в котором счётчик отмечал наибольшее число импульсов. Фотоэлектронный умножитель, а также весь термостат можно было передвигать в вертикальном и поперечном направлениях и вращать в горизонтальной плоскости вокруг зрачка B.

Расположение источников света F_1 и F_2 и точность получения их изображения на фотокатодах не имели особого значения, ибо их роль заключалась лишь в том, чтобы вызывать срабатывание обоих фотоумножителей независимо друг от друга с известной частотой.

Пропускную и отражательную способность зеркала T мы определили с помощью фотометра, калиброванного в микролюменах. Зеркало пропускает 40% падающего на него света и отражает 60%.

§ 5. Регистрирующее устройство

Регистрирующее устройство состояло из фотонных счётчиков, описания которых даны в статьях [3] и [4], и из счётчика совпадений (рис. 4).

Рис. 4. Схема регистрирующего устройства

Усиленные импульсы электронных умножителей M_1 и M_2 с содной стороны подсчитывались, с другой же поступали на счетчик совпадений. Импульсы совпадений регистрировались механическим регистратором, работавшим от управляющего каскада. В электрической схеме счетчика совпадений мы применили пентод, на управляющую и ускоряющую сетки которого поступали приходящие с обеих стор он сигналы.

§ 6. Контрольные измерения

Мы тщательно проверили электронную часть нашего оборудования с помощью методов, описанных в статьях [3] и [4]. Особое внимание уделили мы проверке счетчика совпадений: не пропускает ли он совпадений и не отмечает ли он ложных совпадений. Правда, оборудование было ограждено от внешних помех, но ложные совпадения могут отмечаться и таким путем, что одна сторона нашей установки сработает от импульса электронного умножителя, а усиленный сигнал заставит сработать, в силу ее чрезвычайно высокой чувствительности, и вторую сторону. Чтобы проверить это, мы включили всю электронную часть нашей установки. На одной стороне работал и фотоэлектронный умножитель, а на другой мы прервали питающую цепь электронного умножителя конденсатором большой емкости. Таким образом, на эту сторону могли поступать всякие ложные импульсы, могли возникать ложные совдпаения.

Ложные совпадения нам удалось полностью устранить.

Обе стороны мы настроили так, чтобы при равной интенсивности освещения отсчёты на счётчиках были приблизительно равны.

Важным контрольным измерением явилось установление того, сходится ли разрешающее время, исчисленное на основе формулы (2) из числа случайных совпадений и числа отсчитанных импульсов на обеих сторонах, с непосредственно измеренным разрешающим временем. Последнее замерялось так, что на счётчик совпадений сигналы подавались с одного и того же электронного умножителя, но один из сигналов пропускался через линию задержки. При определенном значении задержки совпадения уже не должны отмечаться. Таким путем мы получили для разрешающего времени значение $\tau = 2{,}3 + 0{,}2 \cdot 10^{-6}$ сек. ($0{,}2 \cdot 10^{-6}$ сек. — интервал перехода). Полученное значение хорошо согласуется со значением, исчисленным из случайных совпадений.

III.

§ 7. Измерения

При измерениях мы через каждые две минуты чередовали когерентное и некогерентное освещение. При каждой замене мы отсчитывали показания счётчиков обоих электронных умножителей (N_1 и N_2) и счётчика совпадений (K). В ходе измерения выявились колебания числа срабатываний. На практике нам удалось получать 20—100 двухминутных периодов отсчёта при неизменном числе срабатываний. Изменение частоты срабатывания было невелико, но было бы всё же затруднительно восстанавливать первоначальные значения после каждого такого изменения. Гораздо целесообразнее

было продолжать измерение, несмотря на изменившуюся интенсивность, и обрабатывать полученные данные таким методом, чтобы сравнение данных, полученных при когерентном и некогерентном освещении, оказалось независимым от изменения частоты срабатывания.

§ 8. Метод обработки наблюдений

Чтобы сделать обработку неблюдений независимой от изменений интенсивности, целесообразно положить в основу обработки разрешающее время τ, — ведь при освещении некогерентным светом возникают лишь случайные совпадения [Формула (2)]. Здесь τ является константой данной установки и не зависит от интенсивности. Число совпадений можно и для случая освещения когерентным светом привести к виду, сходному с формулой (2):

$$K = 2 N^2 \tau^{(k)} t. \tag{8}$$

Из формул (2) и (8) следует:

$$\tau^{(k)} = \tau + \frac{\varepsilon p}{2 N}. \tag{9}$$

Из формулы (9) видно, что, если происходят истинные совпадения, то $\tau^{(k)} > \tau$ — независимо от колебаний частоты срабатываний во время измерения.

Численная обработка полученных данным происходила следующим образом: мы вели расчёт отдельно для каждой серии наблюдений, в пределах которой частота срабатываний оставалась постоянной, т. е. наблюдались только статистические колебания ее. После этого мы сводили результаты, полученные от отдельных серий, в сводный итог.

Данные каждой серии обрабатывались так: из числа срабатываний за каждый двухминутный период как при когерентном, так и при некогерентном освещении мы рассчитывали разрешающее время $\tau^{(i)}$ или — при когерентном освещении — соответствующее ему кажущееся разрешающее время $\tau^{(k)}$. (В дальнейшем индекс (i) относится к данным, полученным при некогерентном освещении, а индекс (k) — к полученным при когерентном освещении.)

$$\tau_\nu^{(k)} = \frac{K_\nu^{(k)} T}{2 N_{1\nu}^{(k)} N_{2\nu}^{(k)}}. \tag{10}$$

(Индексом ν мы различаем данные, полученные за отдельные двухминутные периоды — $T = 120$ сек.) Разумеется,

$$\tau^{(i)} = \tau, \tag{11}$$

т. е. равнозначно действительному разрешающему времени.

Затем мы рассчитали средние значения для каждой серии наблюдений, т. е.:

$$\bar{\tau}^{(k)} = \frac{\tau_1^{(k)} + \tau_2^{(k)} + \ldots + \tau_m^{(k)}}{m},$$
$$\bar{\tau} = \bar{\tau}^{(i)} = \frac{\tau_1 + \tau_2 + \ldots + \tau_m}{m}, \tag{12}$$

где m обозначает число наблюдений в каждой серии. Далее, были рассчитаны средние квадратичные ошибки:

$$\Delta\bar{\tau}^{(k)} = \left[\frac{(\tau_1^{(k)} - \bar{\tau}^{(k)})^2 + (\tau_2^{(k)} - \bar{\tau}^{(k)})^2 + \ldots}{m\,(m-1}\right]^{1/2}$$
$$\Delta\bar{\tau} = \left[\frac{(\tau_1 - \bar{\tau})^2 + (\tau_2 - \bar{\tau})^2 + \ldots}{m\,(m-1)}\right]^{1/2} \tag{13}$$

В случае, если

$$\bar{\tau}^{(k)} - \tau < 3\,[(\Delta\tau^{(k)})^2 + (\Delta\tau)^2]^{1/2} \tag{14}$$

то отклонение $\tau^{(k)}$ от разрешающего времени $\bar{\tau}$ не является существенным и истинных совпадений нет, — вернее, их число не выходит за пределы допустимой погрешности измерений.

Для проверки того, действительно ли колебания в пределах серии наблюдений были только статистическими, мы рассчитали разрешающее время и из среднего числа импульсов K, N_1 и N_2 за серию

$$\bar{\bar{\tau}}^{(k)} = \frac{\bar{K}^{(k)}\,T}{2\,\bar{N}_1^{(k)}\,\bar{N}_2^{(k)}} \quad \text{и} \quad \bar{\bar{\tau}}^{(i)} = \frac{\bar{K}^{(i)}\,T}{2\,\bar{N}_1^{(i)}\,\bar{N}_2^{(i)}}, \tag{15}$$

где $\bar{K}^{(k)} = \dfrac{\sum K_\nu^{(k)}}{m}$ и т. д. ...

Мы рассчитали средние квадратические ошибки рассчитанных таким путем значений $\bar{\bar{\tau}}^{(k)}$ и $\bar{\tau}$:

$$\Delta\bar{\bar{\tau}}^{(k)} = \pm\,\bar{\bar{\tau}}^{(k)}\sqrt{\frac{1}{m\,\bar{K}^{(k)}}} \quad \text{и} \quad \Delta\bar{\bar{\tau}} = \pm\,\bar{\bar{\tau}}\sqrt{\frac{1}{m\,\bar{K}^{(i)}}}. \tag{16}$$

При выводе формулы (16) мы выразили, в соответствии со свойствами распределения Пуассона, среднюю квадратическую ошибку числа совпадений с помощью среднего значения. Для наглядности мы прилагаем таблицу обработки данных короткой серии (таблица I).

Таблица I

Образец обработки серии наблюдений

$T = 120$ сек. $l = 256$

	Когерентные					Некогерентные				
ν	N_1/l	$N_2/$	K	$10^9\,\tau_\nu$	$10^{16}(\Delta_\nu)$	N_1/l	N_2/l	K	$10^9\,\tau_\nu$	$10^{16}(\Delta\tau_\nu)^2$
1	183	173	84	2429	00324	187	182	93	2502	31360
2	179	174	96	2822	11289	185	183	99	2677	01254
3	183	178	94	2642	02433	186	185	107	2847	07952
4	181	181	96	2683	03880	187	186	88	2316	06200
5	179	180	85	2415	00504	185	185	101	2702	01876
6	178	181	93	2643	02464	188	185	84	2211	12532
7	185	181	103	2816	10890	189	186	114	2969	16322
8	178	183	90	2529	00184	186	184	92	2461	01082
9	176	183	92	2615	01664	188	187	77	2005	31360
10	180	186	74	2024	21344	191	189	119	3018	20521
11	180	177	91	2615	01664	187	184	91	2421	02074
12	177	176	81	2381	01102	185	179	110	3041	22658
13	176	181	90	2587	01020	185	183	102	2758	03725
14	178	177	73	2156	10758	194	183	80	2063	25200
15	180	181	65	1827	43428	189	182	89	2369	03842
16	176	179	89	2586	01000	184	184	99	2677	01254
Σ	2866	2781	1396	39772	113948	2996	2947	1545	41037	189212
	$\overline{N_1/l}$	$\overline{N_2/l}$	$\overline{K}$	$\bar{\tau}$	$\Delta\bar{\tau}$	$\overline{N_1/l}$	$\overline{N_2/l}$	$\overline{K}$	$\bar{\tau}$	$\Delta\bar{\tau}$
	179,13	179,44	87,25	2,486 μсек	$\pm$0,069 μсек	187,25	184,19	96,56	8,565 μсек	$\pm$0,088 μсек
				$\Delta\bar{\tau}$	$\Delta\bar{\tau}$				$\bar{\tau}$	$\Delta\bar{\tau}$
				2,485 μсек	$\pm$ 0,065 μсек				2,563 μсек	$\pm$0,062 μсек

§ 9. Результаты измерений

Мы выполнили общим счётом по 119 двухминутных измерений как при когерентном свете, так и при некогерентном. Интенсивность во время измерений была порядка $N = 400$ импульсов в секунду, т. е. $n = 130\,000$ фотонов в сек.

Обработка наблюдений привела к следующим значениям:

при когерентном освещении:
$$\tau^{(k)} = 2{,}484 \pm 0{,}022 \quad \mu\text{сек.}$$
$$\tau^{(k)} = 2{,}484 \pm 0{,}022 \quad \mu\text{сек.}$$

при некогерентном освещении:
$$\tau = 2{,}425 \pm 0{,}022 \quad \mu\text{сек.}$$
$$\tau = 2{,}434 \pm 0{,}022 \quad \mu\text{сек.}$$

Здесь усреднены уже *все* полученные данные.

Видно, что рассчитанные для контроля значения $\tau^{(k)}$ и τ хорошо согласуются соответственно со значениями $\tau^{(k)}$ и τ. Хорошо согласуются и соответствующие ошибки, и это показывает, что колебание числа совпадений в пределах отдельных серий не превышало отклонений, ожидаемых по распределению Пуассона.

Из вышеприведенных данных мы рассчитали ε и $\Delta\varepsilon$. Из формулы (9):

$$\varepsilon = \frac{2N}{p}(\bar{\tau}^{(k)} - \bar{\tau}) \tag{17}$$

и

$$\Delta\varepsilon = \pm \frac{2N}{p}[(\Delta\bar{\tau}^{(k)})^2 + (\Delta\bar{\tau})^2]^{1/2}. \tag{18}$$

Подставив результаты измерений в выражения (9) и (10) вместе со значением $p = 3{,}10^{-3}$ [3], [4], мы получили результат:

$$\varepsilon = 0{,}0076 \pm 0{,}0040.$$

Видно, что значение ε близко к значению ошибки. Это означает, что истинных совпадений не наблюдалось; точнее, что истинные совпадения давало не более 2% фотонов.

IV.

§ 10. Метод частой перемены освещения

В ходе нападки экспериментального оборудования много затруднений доставило обеспечение устойчивости числа срабатываний. В ходе измерений, описанных в разделе III, удалось добиться постоянства частоты срабатывания на протяжении 20—100 двухминутных циклов. Но мы всё же

сочли целесообразным разработать и такой метод, с помощью которого удается совершенно устранить зависимость измерения от колебаний интенсивности освещения. Сущность этого метода заключается в столь частой перемене обоих циклов (освещение то когерентным, то некогерентным светом), чтобы колебаниями освещенности за каждый такой период можно было пренебречь. Чтобы наверняка добиться этой цели, потребовалось

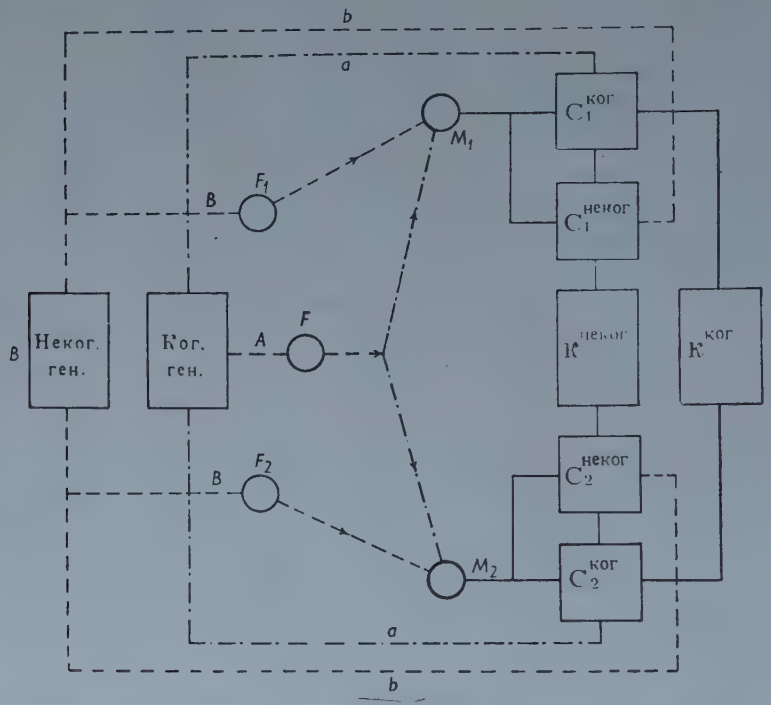

Рис. 5. Схема измерительного устройства, работающего в режиме частой перемены когерентного и некогерентного освещения

переключать освещение по нескольку раз в секунду. Эту задачу мы разрешили с помощью электронного устройства, и с помощью электронного же устройства осуществлялся подсчет с различением импульсов, вызываемых когерентным и некогерентным светом. Работа аппаратуры при быстрой перемене когерентного и некогерентного освещения представлена на рис. 5.

Источник света F, служащий для когерентного освещения, зажигается с помощью генератора A, а служащие для некогерентного освещения источники света F_1 и F_2 — с помощью генератора B. Генераторы работают попеременно, поэтому и лампочки F и F_1, F_2 горят также попеременно. Поступающие с электронных умножителей импульсы и совпадения

мы подсчитываем раздельно для когерентного и некогерентного освещения. Счётчики $C_1^{\text{ког.}}$ и $C_2^{\text{ког.}}$ работают лишь тогда, когда горит лампа F. Этого мы добились с помощью сигнала a, поступающего от генератора для когерентного света. Точно так же включаются с помощью сигнала b счетчики $C_1^{\text{неког.}}$ и $C_2^{\text{неког.}}$ Попеременно же подсчитываются и сигналы совпадений, поступающие при когерентном и некогерентном освещении.

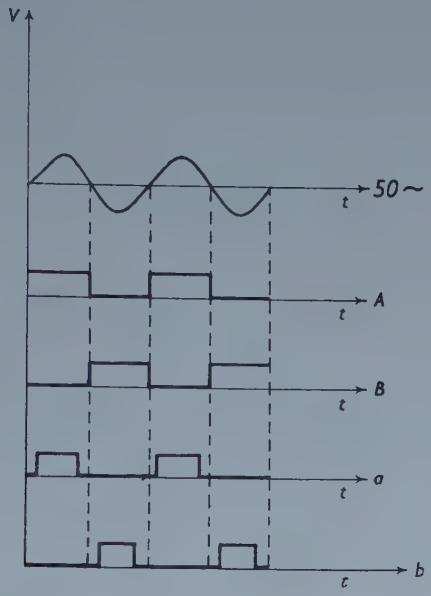

Рис. 6. График сигналов управления и освещения в функции времени

Генератор, служивший для зажигания лампочек и управления счётчиками, был синхронизирован с сетью переменного тока частотой в 50 пер./сек., обеспечивая в течение одного полупериода когерентное, в течение другого полупериода — некогерентное освещение. Сигналы управления электронно-ламповой аппаратурой начинались немного позднее и кончались немного раньше сигналов, зажигающих лампы. График вида сигналов дан на рис. 6.

§ 11. Контрольные измерения

Проверку оборудования мы произвели по методам, описанным в статье [3], и в § 6.

При обработке данных важно знать точную продолжительность сигналов a и b. Не менее важно, чтобы отношение продолжительности этих

сигналов $\left(\frac{a}{b}\right)$ оставалось во время измерения неизменным. Для проверки этого, мы после каждый 5 измерений замеряли значение $\frac{a}{b}$. Это производилось путем освещения электронных умножителей постоянно горящим источником света и подсчета числа срабатываний. Отношение $\frac{a}{b}$ равно отношению чисел срабатываний счётчиков, принадлежащих одному из электронных умножителей и управляемых сигналами a и b. При обработке наблюдений мы использовали среднее значение, полученное из этих измерений. По полученным нами данным, $\frac{a}{b}$ равнялось 0,987.

Сходным способом мы измерили и продолжительность управляющих сигналов. Отсюда $a = 8{,}16 \cdot 10^{-3}$ сек.

§ 12. Результаты измерений

В условиях частопеременного освещения мы произвели 306 трехминутных измерений при интенсивности $N = 300$ импульсов, т. е. $\approx$ 100 000 фотонов в секунду. Трехминутные периоды были нужны нам для статистической обработки наблюдений. Мы получили следующие результаты:

при когерентном освещении:
$$\bar{\tau}^{(k)} = 2{,}362 \pm 0{,}026 \quad \mu\text{сек.}$$
$$\bar{\tau}^{(k)} = 2{,}364 \pm 0{,}030 \quad \mu\text{сек.}$$

при некогерентном освещении:
$$\tau = 2{,}398 \pm 0{,}027 \quad \mu\text{сек.}$$
$$\bar{\tau} = 2{,}405 \pm 0{,}031 \quad \mu\text{сек.}$$

Значения $\Delta\bar{\tau}$ и $\Delta\bar{\tau}$ здесь тоже весьма хорошо сходятся.

Из вышеприведенных данных мы вычислили значения ε и $\Delta\varepsilon$, получив следующий результат:

$$\varepsilon = -0{,}0029 \pm 0{,}0030.$$

Как видим — в сравнении с результатом, приведенным в § 10 — мы получили здесь даже отрицательное значение для ε, так что действительно невозможно приписать ему значение, отличное от нуля.

Мы проводили змерения и при меньшей интенсивности, с частотой срабатывания $N = 42$ импульса в секунду, что соответствует 13 000 фотонов в секунду. В результате 159 трехминутных измерений мы получили для ε и $\Delta\varepsilon$ следующие значения:

$$\varepsilon = -0{,}0017 \pm 0{,}0036.$$

Сравнивая приведенные в настоящем параграфе результаты с результатом, приведенным в § 10, мы получаем из трех серий измерений следующее значение средней квадратичной ошибки ε:

$$\Delta\varepsilon = 0{,}002.$$

Отсюда видно, что истинных совпадений не было; точнее, что истинные совпадения могло дать не более 0,6% всех фотонов. Таким образом, наши результаты сходятся с утверждениями квантовой теории.

Мы хотели бы выразить *Петеру Фараго* и *Жолту Нараи* нашу благодарность за полученные от них ценные советы. Мы должны поблагодарить также наших сотрудников *Шандора Конца* и *Арпада Надь*, участвовавших в проектировании и постройке аппаратуры и в самих измерениях, а также *Ференца Деака*, выполнившего нужные цифровые вычисления.

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COINCIDENCES BETWEEN PHOTONS CONTAINED IN COHERENT LIGHT RAYS

By

A. ÁDÁM, L. JÁNOSSY and P. VARGA

Summary

The authors investigated whether or not photons contained in coherent rays are independent. A light ray falling on a semi-transparent mirror splits into two coherent components. Registering the photons of these components by two electron multipliers, it was investigated, whether or not the electron multipliers do give rise to coincidence discharges. It was shown that making use of an amplifier of usual resolving power, that is, of the order of 2 μsec, the arrangement is suitable for detecting systematic coincidences if such coincidences occur.

In the actual experiment the numbers of coincidences obtained with the multipliers were compared 1) when the multipliers were illuminated by coherent beams, and 2) when they were illuminated by equally strong incoherent beams. The coherent and incoherent illumination was changed in the first experiments in two-minute intervals, in the later experiments by an automatic arrangement at every 1/100 of a second. This fast interchange was employed in order to eliminate as far as possible the effect of slow variations of the sensitivity of the apparatus.

The experiments did not show any difference between the effects of coherent and incoherent light. Thus it can be safely assumed that all the coincidences recorded were accidental coincidences. Moreover, allowing even for fluctuations amounting to three times the standard error, the measurements would be incompatible with an assumption that more than 0,6% of the photons behaved in an anomalous way.

BEMERKUNGEN ZUR ANALYSE DER ABSORPTIONSKURVEN

Von

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(Vorgelegt von P. Gombás. — Eingegangen 1. X. 1954.)

Die Leistungsfähigkeit der von *Doetsch* und *Medgyessy* zur Analyse der Absorptionskurven gegebenen, auf Fourier-Reihen fussenden Näherungsmethode wurde am Spektrum des Fe-Bogenlichtes geprüft. Es konnte festgestellt werden, dass die einmalige Anwendung des Verfahrens im Falle gut gewählter Streuabklingungsparameter sowohl für qualitative (Anzahl der Streifen) wie für quantitative (Wellenlänge) Bestimmungen zum Ziel führt, für die Bestimmung der Intensität dagegen muss das Verfahren wiederholt werden.

Die in der Spektroskopie erhaltenen Absorptionskurven können auf mehrere Weise analysiert werden. Jenes Verfahren, welches die zu analysierende Kurve als Superposition Gausscher Fehlerkurven auffasst und von dieser Grundannahme ausgehend alle (zufällig verborgenen) Komponenten bestimmt, nennt man Gaussche Analyse. Wir wollen uns innerhalb der Gausschen Analyse mit der von *Doetsch* abstammenden sogenannten Streuabklingungsmethode beschäftigen [1], [2], für die *P. Medgyessy* eine zur numerischen Rechnung sehr brauchbare Näherungsmethode ausgearbeitet hat [3], [4].

Die zu analysierende Funktion sei eine Superposition der Gausschen Funktionen der Form

$$\frac{1}{\sqrt{2\pi}\,\sigma_k}\, e^{-\frac{(x-m_k)^2}{2\sigma_k}} \tag{1}$$

und habe die folgende Gestalt

$$f(x) = \sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\,\sigma_k}\, e^{-\frac{(x-m_k)^2}{2\sigma}}, \tag{2}$$

wo m_k die Stelle der Maxima der Gausschen Kurven und σ_k die Streuung ist. (Ihre anschauliche Bedeutung: Halbwertbreite von zwei Wendepunkten der Gausschen Kurve.)

Nehmen wir an, dass wir von $f(x)$ ausgehend die folgende Funktion bestimmt haben:

$$f^*(x) = \sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\sqrt{\sigma_k^2 - \lambda^2}} e^{-\frac{(x-m_k)^2}{2(\sigma_k^2-\lambda^2)}}, \tag{3}$$

wo λ ein geeignet gewählter reeller Wert ist. Infolge des Hineinbauens von λ in die Funktion werden die Gausschen Kurven, welche $f^*(x)$ zusammensetzen, schmäler und höher als die Komponenten von $f(x)$, die Stellen der Maxima bleiben aber dieselben. Wenn man λ so wählt, dass sein Wert kleiner ist als die geringste Streuung der $f(x)$ bildenden Komponenten, dann unterscheiden sich die Komponenten von $f^*(x)$ genügend voneinander um sie zur Bestimmung von spektroskopischen Daten benützen zu können.

Die Frage ist nun, wie man von der Funktion $f(x)$ zu $f^*(x)$ kommen kann. Doetsch arbeitete dafür zuerst eine Methode aus, welche sich aber nur auf den Fall von gleicher Streuung bezog. Seine Methode kann man aber verallgemeinern. Bilden wir die Fourier-Transformierte $\varphi_{(t)}$ bzw. $\varphi^*_{(t)}$ von $f(x)$ bzw. $^*(x)$:

$$\varphi(t) = \int_{-\infty}^{\infty} \left(\sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\,\sigma_k} e^{-\frac{(x-m_k)^2}{2\sigma_k^2}} \right) e^{itx}\, dx, \tag{4}$$

$$\varphi^*(t) = \int_{\infty}^{\infty} \left(\sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\sqrt{\sigma_k^2 - \lambda^2}} e^{-\frac{(x-m_k)^2}{2\sigma_k^2}} \right) e^{itx}\, dx. \tag{5}$$

Die Berechnung dieser Integrale ergibt (siehe [5])

$$\varphi(t) = \sum_{k=1}^{N} A_k e^{im_k t - \frac{\sigma_k^2}{2} t^2}, \tag{6}$$

$$\varphi^*(t) = \sum_{k=1}^{N} A_k e^{im_k t - \frac{(\sigma_k^2-\lambda^2)}{2} t^2} = \varphi(t)\, e^{\frac{\lambda^2}{2} t^2}. \tag{7}$$

Aus (7) erhalten wir wegen der Gültigkeit des Eindeutigkeitssatzes $f^*(x)$, indem wir auf $\varphi^*_{(t)}$ die inverse Fourier-Transformation anwenden. Das von $f(x)$ nach $f^*(x)$ führende Verfahren ist dann das folgende: man bildet die Fourier-Transformierte $\varphi_{(t)}$ von $f(x)$, diese multipliziert man mit $e^{\frac{\lambda^2}{2} t^2}$ und wendet auf diese Funktion die Umkehrung der Fourier-Transformation an.

Diese Methode ist für den praktischen Gebrauch ziemlich schwierig. Im Hinblick auf die uns zur Verfügung stehenden Möglichkeiten für numerisches Rechnen ist jenes Verfahren viel geeigneter, dessen detaillierte Ausarbeitung und Fehlerabschätzung von Medgyessy stammt und welches anstatt mit den in der vorigen Methode auftretenden Fourier-Integralen mit Fourier-Reihen operiert. Das Wesen dieses Verfahrens besteht darin, dass man die im Intervall $(0, l)$ folgendermassen definierten Funktionen

$$f(x) = \begin{cases} \sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\,\sigma_k} e^{-\frac{(x-m_k)^2}{2\sigma_k^2}}, & \text{wenn} \quad 0 \leq x \leq l, \\ 0 & \text{sonst;} \end{cases} \tag{8}$$

$$f^*(x) = \begin{cases} \sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\sqrt{\sigma_k^2 - \lambda^2}} e^{-\frac{(x-m_k)^2}{2(\sigma_k^2-\lambda)^2}}, & \text{wenn} \quad 0 \leq x \leq l, \\ 0 & \text{sonst} \end{cases} \tag{9}$$

im Intervall $(-l, 0)$ gerade fortsetzt und im Intervall $(-l, l)$ die so gewonnenen Funktionen $\overline{f(x)}$ bzw. $\overline{f^*(x)}$ — von welchen man periodische Wiederholung annimmt — in eine Fouriersche cosinus Reihe entwickelt:

$$\overline{f^*(x)} = \frac{Q_0}{2} + \sum_{n=1}^{\infty} Q_n \cos \frac{n\pi}{l} x, \tag{10}$$

wo

$$Q_0 = \frac{1}{l} \int_{-l}^{l} \overline{f(x)}\, dx = \frac{2}{l} \int_{0}^{l} \left(\sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\,\sigma_k} e^{-\frac{(x-m_k)^2}{2\sigma_k^2}} \right) dx, \tag{11}$$

$$Q_n = \frac{1}{l} \int_{-l}^{l} \overline{f(x)} \cos \frac{n\pi}{l} x dx = \frac{2}{l} \int_{0}^{l} \left(\sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\,\sigma_k} e^{-\frac{(x-m_k)^2}{2\sigma_k^2}} \right) \cos \frac{n\pi}{l} x dx, \tag{12}$$

$$\overline{f^*(x)} = \frac{R_0}{2} + \sum_{n=1}^{\infty} R_n \cos \frac{n\pi}{l} x. \tag{13}$$

Hier wiederum ist

$$R_0 = \frac{1}{l} \int_{l}^{l} \overline{f^*(x)}\, dx = \frac{2}{l} \int_{0}^{l} \left(\sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\sqrt{\sigma_k^2 - \lambda^2}} e^{-\frac{(x-m_k)^2}{2(\sigma_k^2-\lambda^2)}} \right) dx, \tag{14}$$

$$R_n = \frac{1}{l} \int_{-l}^{l} \overline{f^*(x)} \cos \frac{n\pi}{l} x dx =$$

$$= \frac{2}{l} \int_{0}^{l} \left(\sum_{k=1}^{N} \frac{A_k}{\sqrt{2\pi}\sqrt{\sigma_k^2 - \lambda^2}} e^{-\frac{(x-m_k)^2}{2(\sigma_k^2-\lambda^2}}, \right) \cos \frac{n\pi}{l} x dx. \qquad (15)$$

Man sieht aus einem Vergleich der in (12) und (15) stehenden Werte von Q_n bzw. R_n mit den (4) bzw. (5) auftretenden Ausdrücken von $\varphi_{(t)}$ bzw. $\varphi^*_{(t)}$ dass man die Werte von Q_n bzw. R_n, abgesehen von einer Konstante mit annähernder Genauigkeit, als reellen Teil der Fourier-Transformierten von $f(x)$ bzw. $f^*(x)$ an der Stelle $t = \frac{n\pi}{l}$ auffassen kann, da die Ausdehnung der Integrationsgrenze ins Unendliche im Ausdruck von Q_n bzw. R_n das Integral nicht wesentlich modifiziert. $f(x)$ ist nämlich nach Definition und also in der Praxis an den Endpunkten des Intervalls klein, ausserhalb des Intervalls aber praktisch Null. So ist nach (7) mit annähernder Genauigkeit:

$$R_n = Q_n e^{\frac{\lambda^2}{2} \frac{n^2\pi^2}{l^2}},$$

d. h. man erhält $\overline{f^*(x)}$ aus $\overline{f(x)}$ (und so $f^*(x)$ aus $f(x)$, wenn man die Fourier-Koeffizienten von $f(x)$ mit $e^{\frac{\lambda^2}{2} \frac{n^2\pi^2}{l^2}}$ multipliziert und mit diesen (13) bildet, natürlich nur bis zu einem gewissen $n = M$. Die ausführlichen Rechnungen teilen wir hier nicht mit, wir geben weiter unten nur die Fehlerschätzung der Näherung an. Da $f^*(x)$ (mit annähernder Genauigkeit) durch den trigonometrischen Reihenschnitt, gebildet mit R_n, gegeben ist, muss zu ihrer Bestimmung eine Fourier-Synthese durchgeführt werden. Weil die kleinste Streuung nicht bekannt ist, muss man im allgemeinen das Verfahren mit etlichen, wachsenden λ Werten wiederholen und von Fall zu Fall untersuchen wie sich die Komponenten trennen.

Unser nächstes Ziel war, die Leistungsfähigkeit der im vorangehenden besprochenen auf Fourier-Reihen beruhenden Näherungsmethode von Medgyessy in der Praxis zu prüfen. Zu diesem Zweck liessen wir von der spektroskopischen Abteilung des Zentralforschungsinstitutes für Physik in Budapest zwei Spektrogramme des Fe-Bogenlichtes aufnehmen mit 20,000 bzw. 200,000-stel Auflösungsvermögen. Wir analysierten das Spektrogramm, aufgenommen mit kleinem Auflösungsvermögen, mit der obigen Methode und verglichen unsere Resultate mit dem Spektrogramm, das mit grossem Ausflösungsvermögen aufgenommen worden war.

Wir verfügten also über die experimentell gefundene Kurve der zu analysierenden Funktion. Von dieser Kurve analysierten wir jene 3 Abschnitte, deren Aufarbeitung am lehrreichsten war. Diese Abschnitte sind durch kontinuierliche Kurven in den Figuren 1a, 2a, und 3a dargestellt.

Bei dem ersten Abschnitt wollten wir untersuchen, ob mittels der Analyse jene 4 Maxima errechnet werden können, welche dem mit «a» bezeichneten scharfen Maximum folgen, bei der Aufarbeitung des 2. und 3. Abschnittes interessierte uns besonders, ob das Verfahren jene 2 Maxima, welche zwischen dem mit «b» und «c» bezeichneten scharfen Maxima liegen, bzw. jenes Maximum, welches zwischen den mit «d» und «f» bezeichneten Maxima liegt, aufweisen kann.

Unsere erste Aufgabe war es, den analytischen Ausdruck der Funktionen aufzuschreiben, welche die einzelnen Abschnitte der Kurve darstellen und zwar der Näherungsmethode gemäss in Fourier-Reihen. Die Fourier-Koeffizienten wurden mit dem Analysator von Mader-Ott bestimmt. In allen drei Fällen waren die höheren Koeffizienten kleiner als der Fehler des Apparates, darum genügte es, 33 Fourier-Koeffizienten zu bestimmen.

	I	II	III
0	3,12	2,94	4,07
1	— 0,34	— 0,51	0,27
2	— 1.19	— 2,18	— 0,99
3	0,68	1,35	0,36
4	— 3,23	— 0,82	— 1,78
5	— 1,04	0,78	— 2,57
6	1,05	1,61	— 1,01
7	0,42	— 1,50	1,26
8	2,04	— 0,32	— 0,78
9	— 0,08	0,46	0,90
10	— 1,04	— 0,45	0,68
11	— 0,43	0,37	1,03
12	— 1,08	— 0,27	0,06
13	0,29	— 0,64	—0,75
14	0,64	0,11	0,11
15	0,49	— 0,08	— 0,53
16	0,53	— 0,13	— 0,12
17	— 0,61	0,04	0,19
18	— 0,54	— 0,13	0,25
19	0,—	0,08	— 0,06
20	— 0,15	— 0,09	0,08
21	0,41	— 0,08	— 0,15
22	0,17	0,10	— 0,18
23	0,11	— 0,05	0,07
24	— 0,17	0,02	— 0,05
25	— 0,24	0,01	0,01
26	— 0,02	— 0,07	— 0,01
27	0,02	0,06	0,08
28	0,13	— 0,02	— 0,01
29	0,07	— 0,01	— 0,01
30	— 0,09	0,—	0,—
31	— 0,13	— 0,04	0,01
32	— 0,07	0,01	0,05
33	0,07	— 0,01	0,06

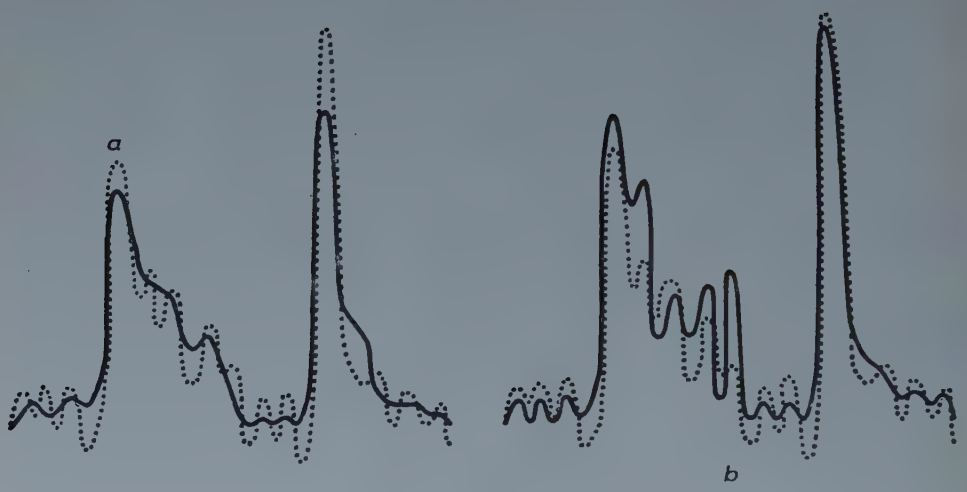

Fig. 1

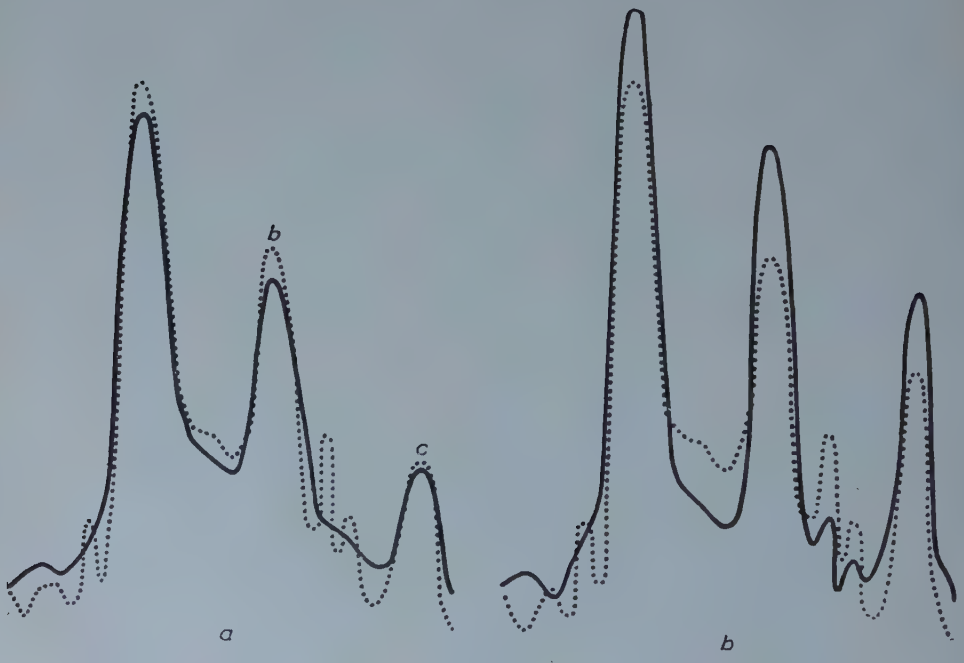

Fig. 2

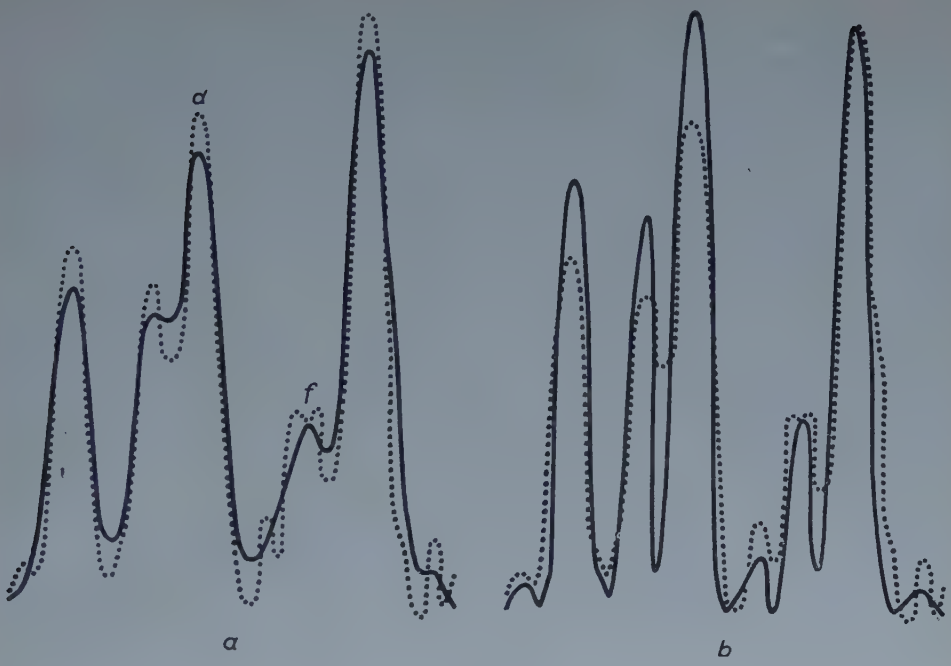

Fig. 3

Bei dem Übergang von Q_n nach R_n wählten wir für den Streuabklingungsparameter den Wert $\lambda = 0{,}447$ ($\lambda^2 = 0{,}2$). Dieser bewährte sich nicht gut, weil sich in $f^*(x)$ sehr grosse negative Komponenten ergaben. Dann rechneten wir mit $\lambda = 0{,}316$ ($\lambda^2 = 0{,}1$). Die punktierte Kurve der Figuren 1a, 2a, und 3a zeigt das Resultat der Synthese.

Um die theoretischen Resultate besser mit den experimentellen Daten vergleichen zu können, zeichneten wir in den Figuren 1b, 2b und 3b die durch Analyse gewonnene Kurve mit einer punktierten Linie und das mit grossem Auflösungsvermögen aufgenommene Spektrum mit einer kontinuierlichen Linie ein. Aus dem Vergleich dieser Kurven kann man sehen, dass im Inneren des Intervalls die analysierte Kurve alle jene Komponenten enthält, welche in dem Spektrum, das mit grossem Auflösungsvermögen aufgenommen war, vorkommen, die Analyse mit Hilfe der Näherungsmethode von Medgyessy konnte also alle Komponenten ausweisen, die uns interessierten. Die Stellen der Maxima der analysierten Kurve im Inneren des Intervalls stimmen ebenfalls genau mit den Stellen des Spektrums, aufgenommen mit grossem Auflösungsvermögen überein. An den Grenzen des Intervalls ist die Übereinstimmung nicht mehr so genau; die Abweichung erklärt sich dadurch, dass man $f(x)$ an den Grenzen des Intervalls, den Bedingungen der Näherungsmethode gemäss, praktisch zu Null abgerundet hat, obwohl doch $f(x)$ an diesen Stellen gewisse kleine Werte hat. Das gute Übereinstimmen der Stellen der Maxima unserer Kurven verleiht

der Methode grosses Gewicht, da man praktisch vor allem die sichere Bestimmung der Stellen der Maxima benötigt (Spektroskopie). Die Form (Stelle der Maxima, Höhe) der Kurve, gewonnen mit der Gausschen Analyse stimmt nicht mit der aufgenommenen Kurve überein, weil die Masstäbe verschieden sind. Man kann aber nachweisen, dass das Verhältnis der Fläche unter den Komponenten (das Intensitätsverhältnis der einzelnen Linien) dasselbe ist, wie das, das man aus der Aufnahme des Spektrogramms bei grossem Auflösungsvermögen des Apparates erhält. Wir wollen darauf hier aber nicht weiter eingehen.

Da man in der Entwicklung in Fourier-Reihen bei der Bestimmung der Koeffizienten nur endliche, und zwar M Glieder, beachtet hat, wurde $f^*(x)$ auch nur in einer gewissen Näherung erhalten, für diesen Fehler gibt Medgyessy die folgende Formel an:

$$|A| < \frac{\left|\sum_{i=1}^{N} A_i\right|}{\min(\sigma_i^2 - \lambda^2)} 2 \left| \frac{2}{\pi} \left| 1 - \Phi\left(\frac{M \min(\sigma_k^2 - \lambda^2)\pi}{l} \right) \right| \right|,$$

wo

$$\Phi(\xi) = \int_{-\infty}^{\xi} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx.$$

Die numerischen Werte für die einzelnen Kurvenabschnitte sind die folgenden, unabhängig von der Stelle:

$$|A_1| < 1{,}05 \qquad |A_2| < 1.31 \qquad |A_3| < 0{,}99.$$

Durch Vergrösserung von M kann man den Fehler weiter herabmindern.

Nach einer Prüfung unserer Resultate kann man also feststellen, dass schon das einmalige Anwenden der Näherungsmethode von Medgyessy mit gut gewählten Streuabklingungsparametern zu qualitativen (Anzahl der Streifen) und quantitativen (Wellenlänge) Bestimmungen gleicherweise geeignet ist: auf Grund der allgemeinen Theorie der Methode bemerken wir hier nur, dass man zur Bestimmung der Intensität das Verfahren in der Weise wiederholen muss, dass man die am meisten herausragende Komponente von der analysierten Kurve graphisch subtrahiert und am Rest die Analyse wiederum ausführt, bis alle Komponenten verzerrungslos erscheinen. Die Gaussche Analyse des Spektrums, das bei kleineren Auflösungsvermögen des Apparates aufgenommen wird, gibt also im grossen und ganzen dasselbe Resultat wie der Apparat bei grösserem Auflösungsvermögen. Dieser Gedanke wurde zuerst von Medgyessy aufgeworfen.

Zum Schluss möchte ich noch folgendes erwähnen. Bei Atomspektren nähert die Dichte-Funktion von Cauchy die Intensitätsverteilung der einzelnen Linien besser an, als die von Gauss. Die Streuabklingungsmethode kann aber nach den Untersuchungen von Medgyessy auch hier angewendet werden.

Das Verfahren beruht auf der Kenntnis des mathematischen Ausdruckes der Intensitätsverteilung. Eine weitere Verbesserung der Methode kann man also nur dann erwarten, wenn die mathematische Gestalt der Intensitätsverteilung genauer bestimmt wird.

Das Verfahren kann schnell nur mit mathematischen Maschinen ausgeführt werden.

Ich danke auch an dieser Stelle aufrichtig Dozenten *J. Horváth*, der mich auf das vorliegende Problem aufmerksam machte und mich beriet. Er ermöglichte die Zusammenarbeit mit dem Institut für Angewandte Mathematik und dem Zentralforschungsinstitut für Physik, diesen sei auch hier mein Dank ausgesprochen, insbesondere für die Zurverfügungstellung des harmonischen Analysators. Ich danke ferner *P. Medgyessy* für seine lehrreichen Hinweise, *T. Mátrai* für die Aufnahme der Spektren, *J. Dombi* für das Registrieren der Spektren und *J. Gyulai* für die Hilfe bei den numerischen Rechnungen.

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ЗАМЕЧАНИЯ К АНАЛИЗУ КРИВЫХ АБСОРБЦИИ

Ф. БЕРЕНЦ

Резюме

Автором была исследована способность приближенного метода с помощью рядов Фурье, данного Дейчем и Медьеши к анализу кривых абсорбции, в случае спектра от Fe-дуги. Было установлено, что применение метода один раз, — в случае хорошо выбранного параметра, уменьшающего разброс, — уже дает годный результат и для качественных (число полос) и для количественных (длина волн) определений, а для определения интенсивности необходимо применять метод повторно.

QUANTUMMECHANICAL FORCES ACTING ON PHOTONS

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By the aid of the scalar representation of the electromagnetic field we construct a statistical ensemble which describes the statistical behaviour of photons. Following D. Bohm's interpretation of quantummechanics we show that here also the motion of a single photon is affected by quantummechanical forces. Finally we call attention to some difficulties of this interpretation.

*

The electromagnetic field free from charges and currents is fully characterized by a vector potential which satisfies the homogeneous wave equation $\Box \mathfrak{A} = 0$ and also the condition $div\ \mathfrak{A} = 0$. *H. S. Green* and *E. Wolf* [1] constructed from $\mathfrak{A}$ a complex scalar wave function V from which inversely the vector potential can be determined. The field equation expressed by V is

$$\Box V = 0 . \tag{1}$$

After substituting $V(x,t) = a(x,t)\, e^{i\Phi(x,t)}$ (a, Φ real), we can readily verify that the equations for a and Φ are

$$\sum_{i=1}^{4} \left(\frac{\partial \Phi}{\partial x_i}\right)^2 - \frac{\Box a}{a} = 0 , \tag{2a}$$

$$\frac{\partial}{\partial x_i}\left(a^2 \frac{\partial \Phi}{\partial x_i}\right) = 0 . \tag{2b}$$

With the help of these equations we can

a) construct a relativistic but classical statistical ensemble, the motion of which is described by (2), as was done by *K. Novobátzky* [2], [3] and *T. Takabayasi* [4], [5] for the case of the Schrödinger and Schrödinger—Gordon equations, and

b) interpret the motion of a single photon as being guided by quantummechanical forces in the sense of *D. Bohm*'s interpretation [6], [7], which origi-

nally refers to electrons only. Here we shall not deal with the general problems of such an interpretation.

First of all we shall determine the quantities describing the statistical ensemble. Introducing $\varphi = \hbar\Phi$ we have, corresponding to (2a) and (2b) respectively:

$$\sum_{i=1}^{4}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 - \hbar^2\,\frac{\Box a}{a} = 0\,, \tag{2a'}$$

$$\frac{\partial}{\partial x_i}\left(\frac{a^2}{\hbar}\,\frac{\partial\varphi}{\partial x_i}\right) = 0\,. \tag{2b'}$$

The latter equation can be regarded as the equation of continuity. Thus we get

$$\alpha\, a^2 \frac{\partial\varphi}{\partial x_i} = \varrho_0\, u_i\,, \tag{3}$$

where ϱ_0 is the density in the rest system, u_i the four-velocity and $\alpha = \text{const}$ After squaring and summing equ. (3) we get, using the identity $\sum_{i=1}^{4} u_i^2 = -c^2$:

$$\varrho_0^2 = -\frac{\alpha^2 a^4}{c^2}\sum_{i=1}^{4}\left(\frac{\partial\varphi}{\partial x_i}\right)^2.$$

Introducing the notation

$$m_0 = \frac{\hbar}{c}\sqrt{-\frac{\Box a}{a}} \tag{4}$$

we get

$$\varrho_0 = \alpha\, a^2\, m_0\,. \tag{5}$$

The numerical value of α can be determined by normalization, its dimension from the dimensions of a, m_0 and ϱ_0. Substituting now (5) into (3) we get

$$u_i = \frac{1}{m_0}\,\frac{\partial\varphi}{\partial x_i}\,. \tag{6}$$

After introducing m_0 we can write (2a') as follows:

$$\sum_{i=1}^{4}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 + m_0\, c^2 = 0, \tag{7}$$

(7) having the form of a Hamilton—Jacobi equation for a particle with the rest mass m_0. Thus interpreting m_0 as a real rest mass, we get; dividing equ. (5) by m_0, the probability density function for a particle:

$$w_0 = a\,a^2. \tag{8}$$

Writing the first characteristic equation of equ. (7) in the following form:

$$p_i = m_0 u_i = \frac{\partial \varphi}{\partial x_i} \tag{9}$$

and eliminating from the second one the $\frac{\partial \varphi}{\partial x_i}$, we get the equation of motion as:

$$\frac{dp_i}{d\tau} = -\hbar c \frac{\partial}{\partial x_i} \sqrt{-\frac{\Box a}{a}}. \tag{10}$$

In the stationary case, where $\varphi = \varphi_0\,(x_1, x_2, x_3) - \delta x_4$ and $\frac{\partial a}{\partial x_4} = 0$ we get in particular

$$E = -ic\,p_4 = ic\delta,$$

where E is the (constant) energy of the particle.

Equations (2b') and (7) allow us to discuss the motion of the photons mutatis mutandis using the interpretation of Bohm. According to this interpretation and to the above equations we can attribute the following characteristic features to a photon.

1. The real rest mass of a photon equals 0.

2. The quantummechanical potential $U(x, t) = \hbar c \sqrt{-\frac{\Box a}{a}}$, which can be determined from the field equations by means of suitable initial and boundary conditions, produces according to the equation $U = m_0 c^2$ a space-time dependent quantummechanical rest mass. (The expression «quantummechanical» means that the respective quantity vanishes for $\hbar \to 0$.)

3. The motion of the photon is described by a usual relativistic equation of motion for a point particle. The force acting on the photon can be determined from the quantummechanical potential.

4. The initial four-momentum of a photon in a space-time point x_0 cannot be chosen arbitrarily; it is $p_i^0 = \left(\frac{\partial \varphi}{\partial x_i}\right)_{x_0}$. If this photon, moving according to equ. (10), arrives in the point x, its four-momentum will be $p_i = \left(\frac{\partial \varphi}{\partial x_i}\right)_x$. 1—4

constitute the main features of the mechanics of photons. The double meaning of $a(x, t)$ (square root of probability density and quantummechanical potential) is acceptable as well under certain conditions (see in [8]). Under the assumption of quantummechanical forces we obtain a statistical theory which has the character of a classical one.

We illustrate our interpretation on two cases.

a) *Circular polarized plane wave.* $A_x = 0$, $A_y = A_0 \cos \omega \left(\frac{x}{c} - t \right)$, $A_z = A_0 \sin \omega \left(\frac{x}{c} - t \right)$. By the help of the transformation described in paper [1] we get: $V(x,t) = A_0 e^{i\omega \left(\frac{x}{c} - t \right)}$, from which $a = A_0$, $\varphi = \hbar\omega \left(\frac{x}{c} - t \right)$. We see from equ. (4) that the «generated» rest mass is zero. The components of the impulse are $p_x = \frac{\hbar\omega}{c}$, $p_y = p_z = 0$, the energy $E = \hbar\omega$, wherever the photon is. The four-velocity is ∞; the photon moves with light velocity in the direction of the positive x axis. The probability of finding the photon is the same everywhere.

b) *Diffraction problem.* Instead of transforming the exact solution of the problem obtained from Maxwell's equations into complex form, we treat this phenomenon in a more simple way. We directly determine the solution of equ. (1) with suitable boundary conditions as superposition of circular polarized plane waves. We use a simple approximation taken from [9] p. 6. According to this solution we get for a diffraction caused by a circular slit of radius about α ($\alpha \gg \lambda$) which stands perpendicular to the x axis:

$$V(x, \varrho, t) = a(x, \varrho)\, e^{i\Phi(x, \varrho, t)} ,$$

where

$$a^2 = \frac{A_0^2\, \alpha^4\, \omega^2}{\omega^2 \alpha^4 + 4 c^2 x^2} \exp\left\{ - \frac{2 \alpha^2 \omega^2 \varrho^2}{\omega^2 \alpha^4 + 4 c^2 x^2} \right\} ,$$

$$\Phi = - \operatorname{arc\,tg} \frac{2 cx}{\omega \alpha^2} + \frac{\omega}{c} x + \frac{2 c \omega \varrho^2 x}{\omega^2 \alpha^4 + 4 c^2 x^2} - \omega t .$$

Here x and ϱ are cylindrical coordinates having their origin in the middle of the slit. After simple calculation we get $p_\varrho \neq 0$; that is, the quantummechanical potential $U \not\equiv 0$ diverts the photons from their original paths along the x axis.

We want now to call attention to some difficulties of the above picture on account of which the former formalism is hardly acceptable as a full description of the relevant phenomena, even if the solution of the wave-corpuscle problem is such as Bohm proposed. Above all we note that from our equations

we cannot infer directly the existence of the spin. In the first case we discussed above e. g. the stream is laminar, i. e. there is no circulation. It seems that the fact that the momentum for a linear polarized wave vanishes is connected as well with the difficulty concerning the spin. The model cannot distinguish between the circular and linear polarized cases according to the spin, it distinguishes them according to the momentum. The «vanishing» of the spin in this model certainly does not follow from the transformation of the field equations to complex form. *P. Román* [10] proved namely that by means of second quantization the existence of the spin follows from the complex field equations too.[1]

The above interpretation is reasonable only in the case $\frac{a}{a} \leq 0$ [see equ. (4)]. The problem of the imaginary rest mass occurs in the analogous interpretation of the Schrödinger—Gordon equation as well. (See [5], p. 199.) Therefore it is not a problem relating to photons only.

Finally we note that equ. (2) allows us to interpret the mechanics of photons also hydrodynamically.

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КВАНТОВО-МЕХАНИЧЕСКИЕ СИЛЫ ДЕЙСТВУЮЩИЕ НА ФОТОНЫ

К. НАДЬ

Резюме

Используя скалярное представление электромагнитного поля автором был построен статистический ансамбль, который описывает статистическое поведение фотонов. Следуя интерпретации квнатовой механики по Д. Бому, видно, что квантово-механические силы здесь тоже оказывают влияние на движение фотона. Наконец, автор обращает внимание на некоторые трудности этой интерпретации.

[1] I would like to thank *G. Marx* for drawing my attention to this fact.

REMARKS ON THE FOUNDATION OF PROBABILITY CALCULUS

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The addition and multiplication axioms of probability calculations are deduced from a number of qualitative assumptions about the way of composition of independent and of excluding events. It is shown that from these assumptions a somewhat generalized addition and multiplication rule can be derived. It turns out, however, that this generalization corresponds only to a possible deformation of scale of probability and thus the generalized scheme does not lead to any results differing from those obtained in the usual way. Furthermore, the ordinary addition and multiplication rules are contained as specialized cases of a more generalized scheme.

§ 1. In the following we start from the conception of probability as a physical quantity like e. g. temperature, energy etc. In a purely qualitative manner there exists a conception of probability: To say something is «very likely» or «very unlikely» has a good meaning. We investigate in the following how it is possible, when starting from this purely qualitative conception of probability, to arrive at the quantitative use of probability and the well-known formalism of probability theory. The problem is somewhat analogous to the following: All investigations of temperature and in the end all thermodynamics took their origin from the fact that one can distinguish between hot and cold. The scientific analysis starting from the purely qualitative distinction between hot and cold gradually led to the discovery of the absolute scale of temperature — thus the scientific analysis succeeded in finding out the objective factor behind the qualitative concepts.

§ 2. Considering problems of probability, I wish to emphasize that in the end we are interested only in very small or very large probabilities. We are interested only in finding out either that a given event is so unlikely that we can safely count on it not to happen — or we are interested in establishing that certain other events have such a high probability to happen, that for all practical purposes we can assume them to be certain. This can be illustrated by examples from all fields where probability calculations are applied: We take as an example the result of a measurement and its error limit. Obtaining a numerical value x as a result of measurement we are interested in the size of the error Δx of the measurement. Thus we are essentially interested in the question, how large Δx must be chosen, so that it is *practically certain* that the error of measurement is less than Δx, thus that the real value of the measured quantity is inside an interval $x + \Delta x$, $x - \Delta x$.

So as to avoid misconceptions from the beginning, we make two remarks.

1. Often one gives as the error of a quantity its standard deviation δx. In that case the probability for the error to exceed δx is by no means small. However, in the latter case the probability that the error exceeds $3\delta x$ is small; if we want to be on the safe side, we can choose as limits of error $4\delta x$ or even $5\delta x$. The probability of the actual error exceeding the latter limits is very small. When we give the standard deviation of a measured quantity we follow a convention and imply that the error is *very likely* less than say $4\delta x$. The convention of giving δx and not say $4\delta x$ as the standard error, has, of course, its good justification as the quantity δx itself plays an important role in the theory of probabilities.

2. We meet statements of the kind that an event has a probability of say 30%. 30% is neither a very small nor a very large probability and thus this seems to contradict our statement at the beginning that we are as a rule interested in very small or large probabilities only. However, if we analyse the statement further, we see that in effect the statement of 30% probability implies also a statement of extreme probability in the following manner. In the case of one event only the fact that this event happens with 30% probability is of little interest. What is of interest is simply this. If we repeat the circumstances of the event very often, say N times, then the number of cases where the event will take place is *almost certain* to be near $n \sim 0{,}3\ N$; similarly one can for practical purposes safely exclude the possibility that the number of events deviates much from $0{,}3\ N$. This means that the statement of 30% probability ultimately refers to certain events which have very high probability (i. e. $n \sim 0{,}3\ N$) and to other cases with very low probability (i. e. $|n - 0{,}3\ N| \gg 1$).

We have given the above example merely as illustration, so as to emphasize our statement about extreme probabilities. The relation $n \sim 0{,}3\ N$ or the corresponding more general one $n \sim pN$ (p probability) shall not be taken for granted in the following considerations, we shall obtain this relation as a result of our analysis.

§ 3. So as to be able to express ourselves conveniently, we shall make use of the following terms. Considering certain types of events, we say that they form a probability field. We use the term in the usual sense. E. g. suppose as events the numbers of counts obtained by a Geiger-Müller counting arrangement during a period of t seconds. For a fixed t the number may be $0, 1, 2, \ldots, k$ (where k is the maximum number the arrangement can count during the time t). Thus our probability field contains $k + 1$ points, representing the possible events.

We shall use the following expression: We *expose* the field, i. e. we switch on the apparatus for a time t. As the result of the *exposition* of the field we get one of the possible events, i. e. one of the points of the field. We ascribe probabilities $p(n)$, $n = 0, 1, 2, \ldots, k$ to the various points of the field. This means that in case of an exposition of the field the result n is obtained with a probability $p(n)$. The probabilities are for the time being introduced in a purely quali-

tative manner. We say that if $p(n) \sim 0$, then it is very unlikely that the event n happens as the result of the exposition. Further we assume that the more likely a result the larger the numerical value of its probability, and for an event which nearly necessarily occurs as the results of an exposition, the probability approaches some limiting value M.

§ 4. *Exclusive events.* To proceed further, we have to classify events. For our purpose it is important to consider *exclusive* events. We call two events contained in the same probability field exclusive, if an exposition of the field can lead only to one event or to the other, but not to both. To go back to our example of counting apparatus, the events where the apparatus has counted during a given interval a specified number of times, are exclusive events. If $k_1 \neq k_2$, then the apparatus, when switched on, may count k_1 or it may count k_2; but, if it counts k_1, then it certainly does not count (during the same exposition) k_2 — and vice versa. It is seen that for events to be exclusive it is of importance to specify them properly. If we were to take as one event that the apparatus has counted anything between 0 and 9, as another event that the apparatus has counted anything between 5—14, then the events thus specified are not exclusive, although we can from the same material also specify exclusive events by taking each number of counts as an event, or by specifying non-overlapping groups as events.

Independent events. Independent events are non-exclusive and in such a way that the happening of the one event does by no means prejudice the happening or not happening of the other event. Events are independent if there is no causal connection between them. It must be emphasized, however, that there is no formal way of establishing for certain that two types of events are independent. Firstly, absolutely independent events hardly exist, thus describing two events as independent we always idealize to some extent. This idealization is, however, necessary. Any successful application of the theory of probability depends on the good idealization of the material. This, by the way, is correct not only for the theory of probability but also for the application of any theory to natural events.

As an important example for independent events we mention the following. We can take the exposition of our counting apparatus (mentioned above) as an event, and we may specify the event by the number of counts obtained. Thus we take as an event the switching on of our apparatus for a time t. Switching on the apparatus several times in succession, we get different events in as much as we obtain different counts. These events can be taken, in a good approximation, as independent events. — More generally the subsequent exposures of a probability field can be taken as events of a more general field and often these events can be assumed to be independent ones.

§ 5. Consider a probability field ; the events contained in the field should be exclusive. Denote the elements of the field with $a, b, c, \ldots$, the corresponding

probabilities by $p, q, r, \ldots$. We can modify the field by changing the classification of the events. Denoting e. g. by A the event which consists in either the event a or the event b, we have a new field containing $A, c; \ldots;$ the probability of A may be denoted P. The latter probability must necessarily be a function of the probabilities p and q, thus we may write

$$P = f(p, q). \tag{1}$$

Thus P is the probability that an exposition of the original field leads either to a or to b. P is also the probability that the modified field in which a and b are joined to the event A, gives the event A as the result of exposure.

We investigate the function $f(p, q)$. We can postulate the following properties of this function :

$$f(p, q) = f(q, p). \tag{2}$$

This relation expresses the fact that no importance can be attached to the order of the events a, b.

Further, if one of the events a, b, say b, cannot occur at all, then we have $q = 0$, and the probability of A is the same as that of a. Thus we may put

$$f(p, 0) = p. \tag{3}$$

We may further assume that A is the more likely, the more likely its components a and b are, thus we can postulate

$$f(p, q) \text{ increases monotonously and is continuous in } p \text{ and } q. \tag{4}$$

We can give another important property of the function f by considering three exclusive events of the field. Denote the event in which a, b or c happens A, and the corresponding probability P. We have at least two possible ways to arrive at the probability P.

1. We modify the original probability field by joining the three events a, b, c to one event A, thus the event A of the new field happens, whenever a or b or c of the original field happen. The probability P is some function of p, q, r, we may write say

$$P = F(p, q, r), \tag{5}$$

where, as before

$$F(p, q, r) = \text{symmetric in } p, q, r. \tag{6}$$

2. We can arrive at the field $A, \ldots$ also in two steps. As the first step we join $a, b \to A$ and arrive at the field $A, c \ldots$. As a second step we join A, c to A and thus arrive at the final field $A, \ldots$. The probability for A or c to happen is obtained according to (1) as

$$P = f(P, r), \tag{7}$$

where

$$P = f(p, q) \tag{8}$$

is the probability for A to happen and r is the probability for c to happen. From (6), (7) and (8) we get

$$f(f(p, q), r) = F(p, q, r) = \text{symmetric in } p, q, r. \tag{9}$$

Thus e. g.

$$f(f(p, q), r) = f(f(r, p), q) = \text{expressions obtained through other permutations in } p, q, r. \tag{10}$$

Equations (1)—(10) give all properties of the function $f(p,q)$ which can be derived on general grounds. In particular, considering groups of more than three events, no further conditions will be imposed on the function $f(p, q)$.

We note that all the above equations are satisfied by the relation

$$f(p, q) = p + q, \tag{11}$$

but they are also satisfied by the more general relation

$$f(p, q) = \psi^{-1}(\psi(p) + \psi(q)), \tag{12}*$$

where $\psi(x)$ is a function with the following properties

$$\psi(0) = 0, \quad \psi(x) \text{ monotonous for } 0 \leqslant x \leqslant M$$

and $\psi^{-1}(y)$ the inverse of $\psi(x)$ exists in the interval

$$0 \leqslant y \leqslant \psi(M)$$

and is also increasing monotonously. Further both $\psi(x)$ and $\psi^{-1}(y)$ must be single valued.

§ 6. With slight modifications we can consider independent events in a way analogous to that used in the preceeding § 5 for exclusive events. Thus consider a field of independent events $a, b, c, d, \ldots$. The probabilities for these events to happen can be written $p, q, r, s, \ldots$. We may now modify the above field by joining the events a, b to A, so that we say $A = a + b$, thus we consider the event A to have happened if during the same exposition of the field

* I am indebted to *A. Békéssy* for drawing my attention to this solution.

both a and b happen. The probability P for A to happen can be composed out of the probabilities p and q as follows :

$$P = g(p, q), \quad p, q \text{ independent.} \tag{1a}$$

We have

$$g(p, q) = g(q, p) \tag{2a}$$

because of lack of order between the events. Further

$$g(p, 0) = 0. \tag{3a}$$

The latter condition implies that, if the event b cannot happen, the combined event $a + b$ must also be excluded.

We have also for the g-function the condition

$$g(M, p) = p\,; \tag{3a'}$$

the latter condition expresses that provided a is certain to happen, the probability of $a + b$ to happen is equal to the probablity of b to happen.

In analogy to (4) we can postulate

$g(p, q)$ increases monotonously with and is continuous in p and q. (4a)

Further we may consider the event $\boldsymbol{A} = a + b + c$ where $\boldsymbol{A}$ happens if all the three independent events a, b and c happen during the same exposition.

Writing for the corresponding probability

$$\boldsymbol{P} = G(p, q, r), \tag{5a}$$

where, as in the case of the F-function, we must postulate for the G-function

$$G(p, q, r) = \text{symmetric in } p, q, r, \tag{6a}$$

grouping again $a + b = A$ and $A + c = \boldsymbol{A}$, we have

$$\boldsymbol{P} = g(P, r) \tag{7a}$$

with

$$P = g(p, q), \tag{8a}$$

thus

$$g(g(p, q), r) = G(p, q, r) = \text{symmetric in } p, q, r \tag{9a}$$

and

$$g(g(p, q), r) = g(g(r, p), q) = \text{expression obtained through other permutations of } p, q, r. \tag{10a}$$

The equations (1a)—(10a) give all possible restrictions on the g-function. All the conditions are compatible with the following

$$g(p, q) = p\,q \tag{11a}$$

with

$$M = 1. \tag{11a'}$$

The restriction (11a') is made necessary by the condition (3a'). The conditions (1a)—(10a) can also be satisfied by the more general expression

$$g(p, q) = \varphi^{-1}(\varphi(p)\,\varphi(q)), \tag{12a}$$

$$\varphi^{-1}(M) = 1, \tag{12a'}$$

where (12a') is made necessary by (3a'). The function φ must obey the same conditions as those we had to impose on the function ψ.

§ 7. We have to consider the connection between the functions f and g. For this purpose it is necessary to consider the combination of exclusive events and independent events. Thus consider a field which contains among others the events a, b, γ,. . ., suppose that a and b are exclusive and that both a and b are independent of γ. The corresponding probabilities may be denoted p, q, ϱ. We consider now the probability that an event

$$A = (a, b) + \gamma \tag{13}$$

occurs ; the event A consists in either a or b happening and also γ happening. Thus we may write for A instead of (13) also

$$A = (a + \gamma,\ \ b + \gamma) \tag{14}$$

where (14) expresses the fact that A consists of either the event $a + \gamma$ i. e. a happening and γ happening, or alternatively of $b + \gamma$ i. e. b happening and γ happening. Denoting the probability for A to happen by Q, we have

$$Q = h(p, q, \varrho),$$

where the function h can be expressed in terms of f and g. Indeed, if we regard A as composed out of a, b, γ according to (13), we have the probability for the event $A = a, b$ to happen

$$P = f(p, q),$$

the probability for the event A and the event γ to happen is thus

$$Q = g(p, \varrho) = g(f(p, q), \varrho). \tag{15}$$

Alternatively, if we consider A as composed of its components according to (14) we may write the following :

$$Q_1 = g(p, \varrho), \qquad Q_2 = g(q, \varrho),$$

where Q_1 is the probability for the event $B_1 = a + \gamma$ to happen, Q_2 the probability for the event $B_2 = b + \gamma$ to happen. Since B_1 and B_2 are exclusive events, we have thus for the probability that either B_1 or B_2 happens

$$Q = f(g(p, \varrho),\ g(q, \varrho)). \tag{16}$$

Comparing (15) and (16) we get a condition connecting f and g, namely :

$$g(f(p, q), \varrho) = f(g(p, \varrho),\ g(q, \varrho)). \tag{17}$$

§ 8. The condition (17) is compatible with the conditions (1)—(10) and (1a)—(10a) as can easily be seen. If we were to choose f and g according to (11) and (11a) together with (11a'), (17) would automatically be fulfilled. If we take the more general forms of f and g as given by (12), (12a) and (12a'), (17) is fulfilled if

$$\varphi(x) = \psi(x); \qquad 0 \leqslant x \leqslant M. \tag{18}$$

§ 9. Summarizing, we find thus, if p and q are the probabilities of two exclusive events a, b, and if further P is the probability for one of the events to happen, we are led to suppose that P is given by

$$P = \psi^{-1}\left(\psi(p) + \psi(q)\right). \tag{19}$$

Further, if a, b are not exclusive but independent events, the probability of both events to happen is

$$P = \psi^{-1}\left(\psi(p)\,\psi(q)\right) \tag{20}$$

with

$$\psi(M) = 1, \tag{21}$$

where $p \sim 0$ expresses unlikely events, $p \sim M$ expresses almost certain events. The function ψ is monotonous and continuous, obeying, apart from (21)

$$\psi(0) = 0, \qquad \psi(p_1) < \psi(p_2) \qquad \text{if} \qquad p_1 < p_2. \tag{22}$$

The equations (19), (20), (21) and (22) are obtained from a number of qualitative postulates about probabilities of groups of events ; these postulates are dealt with in § 5, 6 and 7. The considerations in § 5, 6 and 7 do not lead directly to (19), (20), (21) and (22), but to a number of conditions which are all fulfilled if we postulate those equations ; I have the impression that the latter equations give also the most general relation satisfying the conditions imposed on probabilities.*

The particular assumption

$$\psi(p) = p \tag{23}$$

leads to the usual laws of addition and multiplication of probabilities, so far there appears, however, no necessity to specialize in this way.

§ 10. Let us generalize equations (19), (20) for the case of more than two events.

Suppose thus that the events $a_1, a_2, a_3, \ldots, a_k$ are all exclusive and appear with probabilities $p_1, p_2, p_3, \ldots, p_k$.

The probability P_n that either a_1 or a_2 happens, is given by

$$P_{12} = \psi_{-1}\left[(\psi(p_1) + \psi(p_2)\right]. \tag{24}$$

We call the corresponding event the event A_{12}. The probability P_{123} that either a_1 or a_2 or a_3 happens, is thus the same as the probability that either A_{12} happens or that a_3 happens, thus we have

$$P_{123} = \psi_{-1}\left[\psi(P_{12}) + \psi(p_3)\right], \tag{25}$$

but according to (26) we have

$$\psi(P_{12}) = \psi(p_1) + \psi(p_2).$$

* The problem of functional equation of the type used above was dealt with in detail by *J. Aczé* [1]. In a paper published in the same issue of this journal [2] Aczél has shown that the above impression is indeed correct, provided we assume the functions φ and ψ to be strictly monotonous and continuous. Furthermore, it is shown that not all the conditions imposed upon f and g are necessary. It is e. g. possible to drop some of the conditions imposed upon the probabilities of the independent events without affecting the result.

Therefore

$$P_{123} = \psi_{-1}\left(\psi(p_1) + \psi(p_2) + \psi(p_3)\right). \tag{26}$$

The probability is thus a symmetric function of p_1, p_2, p_3 as it must be.

Extending this procedure to k events, we find the probability $P_{123\ldots k}$ for an event $A_{123\ldots k} = (a_1, a_2\ldots, a_k)$ to happen, given by

$$P_{123\ldots k} = \psi_{-1}\left(\sum_{l=1}^{k} \psi(p_l)\right), \tag{27}$$

where $A_{123\ldots k}$ is that event, where one of the exclusive events $a_1, a_2, \ldots, a_k$ happens.

In a completely analogous manner we can get expressions for the probability for many independent events to happen simultaneously. Consider thus events $b_1, b_2, \ldots, b_k$ which are independent and denote the corresponding probabilities by $q_1, q_2, \ldots, q_k$, then we have

$$Q_{12} = \psi_{-1}\left(\psi(q_1)\,\psi(q_2)\right), \tag{28}$$

where Q_{12} is the probability of the event $B_{12} = b_1 + b_2$, i. e. Q_{12} is the probability of b_1 and b_2 happening at the same exposure. Further we have

$$Q_{123} = \psi_{-1}\left(\psi(Q_{12})\,\psi(q_3)\right),$$

where Q_{123} is the probability for the event $B_{123} = b_1 + b_2 + b_3$ to happen. From (28) we get

$$\psi(Q_{12}) = \psi(q_1)\,\psi(q_2),$$

thus we have

$$Q_{123} = \psi_{-1}\left(\psi(q_1)\,\psi(q_2)\,\psi(q_3)\right).$$

We find further for k events

$$Q_{123\ldots k} = \psi_{-1}\left(\prod_{l=1}^{k} \psi(q_k)\right), \tag{29}$$

where $Q_{123\ldots k}$ is the probability for the event $B_{123\ldots k} = b_1 + b_2 + \ldots b_k$ to happen ; i. e. $Q_{123\ldots k}$ is the probability for all the k events $b_1, b_2, \ldots b_k$ to happen at the same exposure.

We give a further relation. Consider an alternative, i. e. two exclusive events a, $\bar{a}$, one of which is bound to happen at the exposition. The probabilities for a to happen may be denoted by p, the probability for $\bar{a}$ to happen (which is the same as the probability for a not to happen) may be denoted by $\bar{p}$. As one of the events is bound to happen, the probability for a or $\bar{a}$ to happen is the maximum probability M. Thus

$$M = \psi_{-1}\left(\psi(p) + \psi(\bar{p})\right).$$

Applying the function ψ on both sides of the above equation, we find with help of (21)

$$\psi(p) = 1 - \psi(\bar{p}) \qquad \text{or} \qquad p = \psi_{-1}\left(1 - \psi(\bar{p})\right). \tag{30}$$

§ 11. We apply now the above formulae to find the probability in our formalism for the following : Consider a field $a, b, c, \ldots$ of exclusive events with corresponding probabilities $p, q, r, \ldots$. At each exposition of the field we get one of the events of the field. Consider now $N \gg 1$ expositions of the field and suppose that these expositions are independent events. Let us determine the probability that out of the N exposures k give the event a and the remaining $N - k$ expositions give different events. Thus we derive the analogon to the Bernouilli distribution in our modified formalism.

The probability of finding a at one particular exposition is p. According to (29) the probability of getting a k-times subsequently in k expositions is

$$p_k = \psi_{-1}\left([\psi(p)]^k\right).$$

The probability of getting $N - k$ times a result different from a in $N - k$ subsequent expositions is obtained from (29) and (30) as

$$\bar{p}_{N-k} = \psi_{-1}\left([1 - \psi(p)]^{N-k}\right)$$

and the probability for finding a in k specified expositions out of a total of N is thus

$$P_{k,N\ k} = \psi_{\ 1}\left(\psi(p_k)\,\psi(p_{N-k})\right) = \psi_{-1}\left([\psi(p)]^k\,[1 - \psi(p)^{\ N-k}\right). \tag{32}$$

We regard now in the following as one event N subsequent expositions. For sake of distinction we call such an event an «N-fold event». The N-fold event is

characterized by the results of the N expositions of the field. We regard now all those N-fold events which contain exactly k-times the result a. There are $\binom{N}{k}$ such N-fold events; the probability connected with each of these N-fold events is the same, namely $P_{k,\,N-k}$; further these events are exclusive, thus the probability that any of the $\binom{N}{k}$ events, each containing exactly k times the event a, occurs, is according to (27)

$$P_N(k) = \psi_{-1}\left(\binom{N}{k}\psi\left(P_{k,\,N-k}\right)\right)$$

and introducing $P_{k,\,N-k}$ from (32) we have

$$P_N(k) = \psi_{-1}\left(\binom{N}{k}[\psi(p)]^k\ [1-\psi(p)^{N-k}\right). \tag{33}$$

Equation (33) is very similar in form to the Bernouilli distribution.

§ 12. To analyse (33) we determine the most probable value of k for fixed N and p. As both ψ and ψ_{-1} increase monotonously with their argument, we find in the usual way

$$P_N(k) = \text{maximum for } k = k_0$$

with

$$k_0 = [N\,\psi(p)]. \tag{34}$$

(The bracket $[x]$ denotes the largest integer less than x.) For large values of N we may write

$$k_0 \sim N\,\psi(p).$$

Further we determine the probability $P_N(k_1, k_2)$ to find the number of exposures leading to k, k being inside the interval k_1, k_2 $(k_1 < k_2)$. So as to get k exposures leading to a with $k_1 \leqslant k \leqslant k_2$ we can regard the N-fold events corresponding to different k-values in the specified interval as exclusive ones, thus we may write according to (27)

$$P_N(k_1, k_2) = \psi_{-1}\left(\sum_{k_1 \leqslant k \leqslant k_2} \psi\left(P_N(k)\right)\right) \tag{35}$$

and with help of (33) we find

$$P_N(k_1, k_2) = \psi_{-1}\left(\sum_{k_1 \leqslant k \leqslant k_2}\binom{N}{k}[\psi(p)]^k\ [1-\psi(p)]^{N-k}\right). \tag{36}$$

As is well known, the Bernoulli distribution has a sharp maximum at $k = k_0$, such that the sum of the terms around this maximum value is nearly equal to 1. Thus

$$\sum_{k_1 \leqslant k \leqslant k_2} \binom{N}{k} P^k (1 - P)^{N-k} = \Delta_P (k_1, k_2) ,$$

where

$$\Delta_P (k_1, k_2 \sim \begin{cases} 1 \text{ if the interval } k_1, k_2 \text{ is not too narrow, and contains} \\ \quad k_0 \text{ in the middle ;} \\ 0 \text{ if } k_1, k_2 \text{ does not contain } k_0 \text{ and is not too near } k_0. \end{cases} \tag{37}$$

We may thus write for (36)

$$P_N (k_1, k_2) = \psi_{-1} \left(\Delta_{\psi (p)} (k_1, k_2) \right) ,$$

and thus we find

$$P_N (k_1, k_2) \sim \left\{ \begin{matrix} M \\ 0 \end{matrix} \right\} \text{ conditions as in (37).} \tag{38}$$

We can express our result thus : Among N exposures we shall almost certainly find k expositions leading to the event a, where k is a number in the vicinity of $N \psi(p)$. As in the ordinary treatment we can show e. g. that with increasing N the probability tends towards zero for k to fall outside

$$N \left(\psi (p) \pm \varepsilon \right) ,$$

where ε is an arbitrarily small quantity.

§ 13. We are now in a position to introduce the ordinary rules of probability calculation. We saw that from purely qualitative assumptions, which themselves are nearly trivial, we could derive the result that repeating an exposure a very large number N of times, an event a of the exposed field will nearly certainly appear about

$$k \approx N \psi (p) \tag{39}$$

times ; N is the number of repetitions and p the qualitatively defined probability of the single event. The important feature of this result is that it shows

$$k \backsim N \text{ for large } N.$$

Thus, without having introduced any quantitative definition of the probability, we could show by the purely qualitative considerations that the number of events leading to a increases about proportional to N. Expressing this result more precisely we say that the ratio

$$P_N = \frac{k_N}{N}$$

with increasing N has a tendency to approach a constant value. k_N is the actual number of results leading to a. This «tendency» can be formulated in different ways, e. g. by saying that for any arbitrarily small $\varepsilon > 0$ the relation

$$|P_N - P_M| < \varepsilon$$

is almost certainly fulfilled for fixed N and M provided $1 \ll N \ll M$.

Of course, we have to treat the above equation carefully, if we determine successively differences

$$|P_N - P_{M_1}|, \quad |P_N - P_{M_2}|, \quad \ldots \text{ etc.},$$

thus after a sufficiently large number of such determinations, it is almost certain that we shall find cases where the difference is not small but large. Therefore another formulation of the case is preferable : The value of k_N/N with increasing N shows a tendency to fluctuate more and more closely around a fixed value. This fluctuation is of course such that any of the actual values in the series may deviate even by large amounts from the fixed value around which the fluctuation is taking place. However, the larger N the smaller the probability of a large deviation — and for a sufficiently large value of N we can for practical purposes take k_N/N as a good approximation of $\psi(p)$ the value around which the ratio fluctuates. The risk involved in putting

$$\psi(p) \approx k_N/N$$

for sufficiently large N is exactly of the same type as the risk generally involved when relying on the results of any physical measurement.

§ 14. So far we could not fix the function $\psi(p)$ itself. From the foregoing considerations it appears that all essential results of probability theory can be obtained *without* specifying $\psi(p)$, in fact the results are completely independent of the actual choice of $\psi(p)$ as we have shown in the case of the Bernouilli distribution. It is therefore admissible and at the same time useful to make the simplest choice of $\psi(p)$, compatible with the conditions imposed in § 5, namely

$$\psi(p) = \psi_{-1}(p) = p \quad \text{and} \quad M = 1. \tag{40}$$

($M = 1$ follows necessarily from the choice of $\psi(p)$ since $\psi(M) = 1$ according to (21).)

Assuming (40), we have

$$p \approx k_N/N \tag{41}$$

or

$$k \approx pN.$$

The latter relations must by no means be regarded as definition of probability. We have shown from purely qualitative considerations, without assuming anything of the kind of equation (41), that the right hand side of (41) fluctuates around a fixed value, the latter being a monotonous function of the probability p. By postulating (40) we do nothing else but fix quantitatively the scale of probabilities. We may express this also in the following manner.

From general considerations, without introducing any quantitative scale for probabilities we come to the conclusion that the number of cases k_N is about proportional to N («about» is to be understood as explained further above). k_N will be in general the larger, the larger the probability p of the event a. We are free to postulate that the proportionality constant is *equal* to the probability. This postulate merely fixes the quantitative *scale* of probabilities, without effecting any essential result of the theory.

Thus postulating in this sense (40), we have

$$P = \sum p_k \quad \text{instead of} \quad P = \psi_{-1}\left(\sum \psi(p_k)\right),$$

$$Q = \prod q_k \quad \text{«} \quad \text{«} \quad Q = \psi_{-1}\left(\prod \psi(q_k)\right),$$

thus we get the ordinary rules of probability calculation.

§ 15. The above considerations were inspired by a paper of *Schrödinger* [3], although our considerations are not identical with those in the latter paper.

A. Rényi has drawn my attention to the fact that *Bernstein*, 1917, has given considerations somewhat similar to those given here. I was unable to obtain Bernstein's original paper [4], however, judging from Bernstein's book [5] on the theory of probability (Moscow, 1946) Bernstein's considerations, although showing a remarkable similarity to our considerations, are by no means identical.

We discuss shortly the connection between our considerations explained above and the well-known axiomatic system of probability theory given by *Kolmogorof*. We note first that there is complete agreement between our considerations and those of Kolmogorof. In fact our considerations complete in a certain sense Kolmogorof's axiomatic theory. While Kolmogorof introduces addition and

multiplication of probabilities as axiom connected with properties of sets, we give considerations which justify these axioms from purely physical considerations, at least in the case of discrete probability distributions.

To see these connections in some more detail, we sketch very briefly Kolmogorof's system and point out the connections.

Kolmogorof considers a set U, the elements of which may be called elementary events. These elementary events can be regarded as what we have described in this article «exposures of a field». Further, a set F is considered, the elements of which are all the sub-sets of U. The properties of F are as follows.

1. F should contain U as element,

2. if A and B are sub-sets of U, then $A + B$ and $A \cdot B$ and also $\bar{A}$ and $\bar{B}$, should be elements of F,

3. if $A_1, A_2, \ldots, A_n, \ldots$ are elements of U, then $A_1 + A_2 + \ldots + A_n + \ldots$ and $A_1 \cdot A_2 \cdot \ldots A_n \ldots$ should also be elements of F. The elements of F play the role of the statistical events.

1. If among the elements of the sets A and B are no common elements, then we denote the events corresponding to A and B exclusive events. Further, we call the U-events *certain* events, and the U-events (empty set) *excluded* events.

To every set A a non-negative number $P(A)$ corresponds and this number is called the probability attached to the event A.

The following axioms are postulated for the probabilities thus obtained:

1. $0 \leqslant P(A) \leqslant 1$.

2. $P(U) = 1$.

3. If A and B are mutually exclusive events, then

$$P(A + B) = P(A) + P(B).$$

Instead of the above axioms we have in our considerations further above the functional relation (12).

2. If in a series of events we are restricting ourselves to those events A which can only occur provided one of the events B has occurred, then we use for the events A the symbol A/B. We call the latter kind of events *conditional* events and make to correspond to these conditional events non-negative numbers $P(A/B)$ which we call conditional probabilities.

We consider the series of events belonging to the sets A and B and form out of the conditional events a new set F. The elements of this new set are those events where simultaneously, one event each, of the set A and the set B has occurred. We make to correspond again to this latter series of events non-negative

numbers denoted by $P(A \cdot B)$. It is easy to show that for two events A and B the following relation exists:

$$P(A \cdot B) = P(B)\, P(A/B).$$

We call the events A and B *independent* events provided we have

$$P(A) = P(A/B).$$

In our considerations the above equations are replaced by the equation (12a), which appears in a somewhat more general form, but has essentially the same contents as Kolmogorof's relation.

I am greatly indebted to *L. Pál* for discussions and in particular for helping to clarify the connection between the contents of the above article and Kolmogorof's theory.

Note added in proof. After the manuscript of the above paper was submitted for publication, *A. Rényi* drew my attention to *I. J. Good*'s book »Probability and the Weighing of Evidence», Charles Griffin & Co., London 1950. In an Appendix (page 105) J. Good sketches ideas which seem to be nearly identical with those worked out in the present paper in some more detail. It is interesting that Good refers also to Schrödinger in his Appendix.

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ЗАМЕЧАНИЯ ОТНОСЯЩИЕСЯ К ОСНОВАМ ТЕОРИИ ВЕРОЯТНОСТИ

Л. ЯНОШИ

Р е з ю м е

Аксиомы сложения и умножения теории вероятности были выведены из нескольких качественных предположений, связанных с независимыми и исключающими друг друга событиями.

Покажем, что из этих предположений могут быть установлены обобщенные, до некоторой степени, теоремы сложения и умножения. Однако оказывается, что это обобщение соответствует лишь одной возможной деформации шкалы вероятностей, и таким образом, результаты, полученные на основе обобщенной схемы, не отличаются от результатов, полученных обычным путем. Далее, обычные теоремы сложения и умножения содержатся как специальные случаи более общих схем.

A SOLUTION OF SOME PROBLEMS OF K. BORSUK AND L. JÁNOSSY

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The aim of the present paper is the determination of those continuous strictly monotonic and associative operations for which addition (problem of *A. Gruzewski—K. Borsuk*), multiplication or the quasimultiplication $\psi^{-1}[\psi(x)\,\psi(y)]$ are distributive, further the determination of those operations which are distributive for addition or the quasi-addition $\psi^{-1}[\psi(x)+\psi(y)]$. The respective operations are ${}^{a}\log(a^x+a^y)$, $\sqrt[k]{x^k+y^k}$, $\psi^{-1}\left[\sqrt[k]{\psi(x)^k+\psi(y)^k}\right]$, cxy, $\psi^{-1}[c\psi(x)\psi(y)]$. From this we derive that the probability of the occurrence of one of two exclusive events resp. of both of two independent events has the form $\psi^{-1}[\psi(p)+\psi(q)]$ resp. $\psi^{-1}[\psi(p)\,\psi(q)]$ as was suggested by *Jánossy* in the preceding paper if we suppose only that the probability of (A *or* B) increases continuously with the probabilities of A and B, and the probability of the happening of one of three events does not depend on their arragement, further the non-negative probability of the simultaneous occurence of two events does not depend on their order and finally the probability of (A *or* B) *and* C equals that of (A *and* C) *or* (B *and* C). On the other hand by modifying a theorem of S. *Golab* we see that the same formulas are valid (the second even for not-independent events) if besides this last condition these probability functions are derivable (the second with respect to its second variable in a continuous manner) the first being also strictly monotonic and if the probability of A *or* the impossible event resp. of A *and* the certain event (without regard to their order) is equal to the probability of A, and finally the simultaneous occurrence of any event with the impossible event is also impossible.

Further similar statements are also proved.

§ 1

In his paper [7] appearing in this issue *L. Jánossy* suggests in § 9 that his postulates for probability:

$$0 \leq p, q, f(p,q),\ g(p,q) \leq M;$$

$$f(p,q) = f(q,p), \qquad g(p,q) = g(q,p);$$

$$f(p,0) = p, \qquad g(M,p) = p, \qquad g(p,0) = 0;$$

$f(p,q)$ and $g(p,q)$ continuously increasing with p and q

$$f[f(p,q),r] = f[p,f(q,r)] = \ldots; \quad g[g(p,q),r] = g[p,g(q,r)] = \ldots$$

and

$$g[f(p,q),r] = f[g(p,r),g(q,r)]$$

imply

$$f(p, q) = \psi^{-1}[\psi(p) + \psi(q)], \tag{1}$$

$$g(p, q) = \psi^{-1}[\psi(p) \cdot \psi(q)]. \tag{2}$$

($f(p, q)$ resp. $g(p, q)$ are the probabilities for the occurrence of one of two exclusive resp. of both of two independent events with the probabilities p and q.)

We will show that already a few of these properties are sufficient for the characterization of the functions (1) and (2). In §§ 3,5 and 6 we prove three theorems containing three such systems of postulates. This will be facilitated by investigating the more direct questions, as to those operations which are distributive with respect to addition and those which are distributive with respect to multiplication. An analogous question in regard to operations with respect to which addition is distributive was raised by *A. Gruzewski* and *K. Borsuk* (oral communication of *K. Borsuk*). §§ 2, 4 and 6 contain the answers to these questions.

Our problems are related to those investigated by *M. Hosszu* in his paper on distributivity [6]. Here we will, however, (besides other special features) in theorems 1—5 at most suppose continuity, while *Hosszu* needs derivability for his solution. (Similar problems which are nearer to our special case, though also with conditions of derivability were treated by *S. Golab* [5], see § 6.)

§ 2

A. Gruzewski was interested in the question as to which operations $F(x, y)$ can be called «pre-additions», in the sense that addition is distributive with respect to these operations:

$$F(x, y) + z = F(x + z, y + z). \tag{3}$$

In this sense every $f(x, y) = x + \varphi(y - x)$ would do, or if we want also commutativity (symmetry) then $f(x, y) = \frac{x + y}{2} + \varphi(x - y)$. Thus the problem modified by *Borsuk* is to find all *associative* (continuous) operations satisfying (3) and ordering to real numbers of an interval real numbers of the same interval. It might be verified, that

$$F(x, y) = {}^{a}\log(a^{x} + a^{y})$$

satisfies these conditions.

If $F(x, y)$ is supposed to be increasing [2] or what is equivalent ([3], [10]) the cancellation law [$F(x, t) \equiv F(x, u)$ resp. $F(t, y) \equiv F(u, y)$ implies $t = u$] holds, then the associativity

$$F[F(x, y), z] = F[x, F(y, z)] \tag{4}$$

implies the existence of a strictly monotonic and continuous function φ such that

$$F(x, y) = \varphi^{-1}[\varphi(x) + \varphi(y)]. \tag{5}$$

This function φ is determined but for a constant factor, that is

$$\varphi^{-1}[\varphi(x) + \varphi(y)] = \psi^{-1}[\psi(x) + \psi(y)] \tag{6}$$

if and only if

$$\psi(x) = a\varphi(x).$$

(See e. g. [1]). If we substitute (5) into (3), we have

$$\varphi^{-1}[\varphi(x) + \varphi(y)] = \psi^{-1}[\psi(x + z) + \psi(y + z)] - z.$$

By keeping z constant we see that this is an equation of the form (6) with $\psi(x) = \varphi(x + z)$ and thus

$$\varphi(x + z) = a(z)\,\varphi(x), \tag{7}$$

where together with $\varphi(z)$ also $a(z)$ is continuous. By putting $x = 0$, $\varphi(0) = \alpha$ we get [9]

$$\varphi(z) = \alpha\, a(z)$$

$$\alpha\, a(x + z) = \alpha\, a(x)\, a(z), \quad a(x + z) = a(x)\, a(z)$$

and this is an equation of the Cauchy-type [4],

$$a(z) = a^z, \qquad \varphi(z) = \alpha a^z,$$

$$F(x, y) = \varphi^{-1}[\varphi(x) + \varphi(y)] = {}^{a}\log(a^x + a^y).$$

Thus we have the

Theorem 1. If $F(x, y)$ is associative, continuous and increasing (resp. the cancellation law holds) and if the distributivity-equation (3) *is satisfied then and only then*

$$F(x, y) = {}^{a}\log(a^x + a^y).$$

An equivalent of this is the

Theorem 2. If $F(x, y)$ is associative, continuous and increasing (resp. the cancellation law holds) and if multiplication is distributive with respect to this operation i. e.

$$F(x, y)\, z = F(xz, yz) \tag{8}$$

(homogeneity with exponent 1) then and only then

$$F(x, y) = \sqrt[k]{x^k + y^k}\,.$$

The problem to which this theorem gives an aswer is yet more natural, as it asks for all operations with respect to which multiplication is distributive. $k = 1$ gives of course the addition.

Theorem 2 can be reduced by logarithmisation to theorem 1, but because of the importance of this theorem for the problem of L. Jánossy we give a direct proof, this being of course a variant of the proof of theorem 1.

We have again from the associativity continuity and monotony

$$F(x, y) = \varphi^{-1}[\varphi(x) + \varphi(y)]$$

and from (8):

$$\varphi^{-1}[\varphi(x) + \varphi(y)] = \frac{\varphi^{-1}[\varphi(xz) + \varphi(yz)]}{z}.$$

This is again an equation of type (6) with $\psi(x) = \varphi(xz)$ and thus

$$\varphi(xz) = a(z)\,\varphi(x),$$

the solution of which can be reduced in the same way as (7) to another Cauchy-type equation

$$a(xz) = a(x)\,a(z).$$

So finally

$$\varphi(z) = a\,z^k,\quad F(x, y) = \sqrt[k]{x^k + y^k},$$

q. e. d.

§ 3

Theorem 2 helps us to find a first answer to Jánossy's problem:

Theorem 3. If $f(p, q)$ and $g(p, r)$ are continuous, increasing and associative and g is distributive from the right with respect to f:

$$g[f(p, q), r] = f[g(p, r), g(q, r)], \tag{9}$$

then and only then there exists a continuous and strictly monotonic function $\psi(p)$ such that

$$f(p, q) = \psi^{-1}[\psi(p) + \psi(q)],\qquad g(p, r) = \psi^{-1}[\psi(p)\,\psi(r)].$$

If

$$f(p, o) = p,\quad g(p, M) = p,\quad \textit{then}\quad \psi(o) = 0,\quad \psi(M) = 1.$$

Proof: Because of the assumptions for g it is

$$g(p, r) = \varphi^{-1}[\varphi(p) + \varphi(r)]$$

or with the notation $\Phi(p) = e^{\varphi(p)}$

$$g(p, r) = \Phi^{-1}[\Phi(p) \cdot \Phi(r)]. \tag{10}$$

By substituting this in (9) we have

$$\Phi^{-1}\{\Phi\ [f(p,q)]\cdot\Phi(r)\}=f\{\Phi^{-1}[\Phi(p)\,\Phi(r)],\ \Phi^{-1}[\Phi(q)\,\Phi(r)]$$

or by introducing the notations

$$x=\Phi(p),\ \ y=\Phi(q),\ \ z=\Phi(r),\ \ F(x,y)=\Phi\{f[\Phi^{-1}(x),\Phi^{-1}(y)]\}$$

we get

$$F(x,y)\,z=F(xz,yz),$$

but this is equation (8). As $F(x,y)$ together with $f(p,q)$ is associative, increasing and continuous, theorem 2 gives

$$F(x,y)=\sqrt[k]{x^k+y^k},$$

that is

$$f(p,q)=\Phi^{-1}\{F[\Phi(p),\Phi(q)]\}=\Phi^{-1}\left[\sqrt[k]{\Phi(p)^k+\Phi(q)^k}\right]$$

or with $\Phi(p)^k=\psi(p)$

$$f(p,q)=\psi^{-1}[\psi(p)+\psi(q)].$$

On the other hand (10) can be written

$$g(p,r)=\Phi^{-1}\left[\sqrt[k]{\Phi(p)^k\,\Phi(q)^k}\right]=\psi^{-1}[\psi(p)\cdot\psi(p)].$$

q. e. d.

§ 4

We obtain a stronger theorem by reversing the problem of theorem 2, that is by looking for operations which are distributive with respect to addition. These are ruled by

Theorem 4. All functions $G(x,z)$ non-negative in a positive interval (or monotonic or continuous or measurable or bounded from one side, etc.) which are distributive from the right with respect to addition :

$$G(x+y,z)=G(x,z)+G(y,z) \tag{11}$$

are of the form

$$G(x,z)=c(z)\,x, \tag{12}$$

and if $G(x,z)$ is also symmetric (commutative) at least for a $z=a$: $G(x,a)=G(a,x)$ (or distributive with respect to addition also from the left) then and only then

$$G(x,z)=cxz.$$

The first statement is an evident consequence of the fact ([8]) that the most general non-negative (or monotonic etc.) solution of the Cauchy-equation

$$f(x+y) = f(x) + f(y)$$

is

$$f(x) = cx.$$

On the other hand if G is also commutative for $z = a$, then (12) gives

$$c(x)\, a = G(x, a) = G(a, x) = c(a)\, x, \quad \frac{c(x)}{x} = \frac{c(a)}{a} = c \qquad \text{(constant)}$$

i. e.

$$G(x, z) = cxz,$$

q. e. d.

(If instead of commutativity, distributivity on the other side is supposed, then the same follows from

$$G(x, z) = c(z)\, x = d(x)\, z, \qquad \frac{c(z)}{z} = \frac{d(x)}{x} = \text{constant.})$$

§ 5

Our second,somewhat stronger answer to Jánossy's problem is contained in the following

Theorem 5. If $f(p, q)$ is continuous increasing and associative and $g(p, r)$ is non-negative commutative [at least for one r, e. g. $r = M$: $g(p, M) = g(M, p)$] and (9) *distributive from the right with respect to $f(p, q)$, then and only then*

$$f(p, q) = \psi^{-1}[\psi(p) + \psi(q)], \quad g(p, r) = \psi^{-1}[\psi(p) \cdot \psi(r)].$$

The commutativity of g can be replaced by the distributivity from the left side. — Also here $\psi(o) = 0$, $\psi(M) = 1$, $[f(p, o) = p,\ g(p, M) = p]$.

Proof: The assumptions for f imply

$$f(p, q) = \varphi^{-1}[\varphi(p) + \varphi(q)]. \qquad (5')$$

By substituting this into (9) we have

$$g\{\varphi^{-1}[\varphi(p) + \varphi(q)],\ r\} = \varphi^{-1}\{\varphi[g(p, r)] + \varphi[g(q, r)]\}$$

or with the notations

$$x = \varphi(p),\ y = \varphi(q),\ z = \varphi(r),\ \varphi\{g[\varphi^{-1}(x), \varphi^{-1}(z)]\} = G(x, z):$$
$$G(x + y, z) = G(x, z) + G(y, z)$$

and thus from theorem 4 [as $G(x, a) = G(a, x)$ for $a = \varphi(M)$ from $g(p, M) = g(M, p)$:

$$G(x, z) = cxz\ ,$$

$$g(p, r) = \varphi^{-1}[c\varphi(p)\varphi(r)] = \varphi^{-1}\left[\frac{c\varphi(p) \cdot c\varphi(r)}{c}\right]$$

or with $\psi(p) = c\varphi(p)$

$$g(p, r) = \psi^{-1}[\psi(p)\psi(r)].$$

At the same time (5′) can be written also

$$f(p, q) = \varphi^{-1}\left[\frac{c\varphi(p) + c\varphi(q)}{c}\right] = \psi^{-1}[\psi(p) + \psi(q)],$$

q. e. d.

Theorems 3 and 5 can be interpreted as follows: The probabilities of the occurrence of one of two exclusive events and of both of two independent events can be expressed by the probabilities p, q of the original events by formulas (1) and (2); and only if one of the following two systems of conditions is satisfied: A) These two composed probabilities are continuous and increasing with p and q, the probability of the happening of one of three exclusive events and of all the three independent events does not depend on how these events are arranged and finally the probability of the events *A or B and* the event *C* is equal to that of the events (*A and C*) *or* (*B and C*). Or B) the probability of (*A or B*) increases continuously with the probabilities of *A* and *B* and the probability of the occurrence of one of three events does not depend on how they are arranged, further the probability of the simultaneous occurrence of two independent events is non-negative and does not depend on the order of the events (one of wich might be the certain event) and finally again the probability of (*A or B*) *and C* equals that of (*A and C*) *or* (*B and C*).

The second system of conditions might be preferred because it postulates less about the more difficult concept of independent events and can be modified, as we shall see in the next §, so that independence needs not to be postulated at all.

§ 6

We see that our theorems are all connected with the questions: which are the operations which are distributive with respect to addition or «quasi-addition» (1) or which are the operations with respect to which addition or

multiplication or the «quasi-multiplication» (2) is distributive. Our results can be condensed in the typical distributive pair

$$f(x,y) = \sqrt[k]{x^k + y^k}, \qquad g(x,y) = cxy.$$

Our theorems are strongly connected. Not only theorems 1, 2, 3 are related to each other and similarly theorems 4 and 5 but also e. g. theorem 3 can be derived — as one sees immediately — also from theorem 4 instead of from theorem 2. Moreover theorem 1 (and similarly theorem 2) can also be proved in an analogous way as theorems 4 and 5 in the following manner:

$F(x,y)$ being continuous, increasing and associative we have again

$$F(x,y) = \varphi^{-1}[\varphi(x) + \varphi(y)]$$

and with help of (3)

$$\varphi^{-1}[\varphi(x) + \varphi(y)] + z = \varphi^{-1}[\varphi(x+z) + \varphi(y+z)]$$

or with the notations $\xi = \varphi(x)$, $\eta = \varphi(y)$, $\zeta = \varphi(z)$,

$$G(\xi,\zeta) = \varphi[\varphi^{-1}(\xi) + \varphi^{-1}(\zeta)] \text{ (commutative!)}$$

$$G(\xi+\eta,\zeta) = G(\xi,\zeta) + G(\eta,\zeta),$$

but this is equation (11) and so with theorem 4

$$G(\xi,\zeta) = c\xi\zeta = \frac{1}{c} e^{\ln c\,\xi + \ln c\,\zeta} = \varphi[\varphi^{-1}(\xi) + \varphi^{-1}(\zeta).]$$

This is an equation of the type (6) and thus

$$\varphi^{-1}(\xi) = a \ln c\,\xi, \qquad \varphi(x) = \frac{1}{c} e^{\frac{x}{a}} = \beta a^x$$

$$F(x,y) = \frac{a}{\log}(a^x + a^y),$$

q. e. d.

The proof of theorem 2 goes on the same lines.

A theorem which characterizes the solutions

$$f(p,q) = \psi^{-1}[\psi(p) + \psi(q)], \tag{1}$$

$$g(p,q) = \psi^{-1}[\psi(p)\,\psi(q)] \tag{2}$$

of

$$g[f(p,q),r] = f[g(p,r),g(q,r)] \tag{9}$$

without privileging either of them, but which supposes also derivability, was proved — as we have already mentioned — by *S. Golab* [5]. He proved that, if in (9) f and g are continuously derivable,

$$f(p, o) = f(o, p) = p, \qquad g(M, p) = p, \qquad g(o, p) = g(p, o) = 0$$

are satisfied and also $\frac{\partial^2}{\partial p \partial q} g(p, o)$ exists and $f(b, p) = c$ can be solved for p with *every* pair b, c, then f and g are of the form (1), (2). As for our purposes the last condition (the solvability of $f(b, p) = c$) is not suitable (it would involve probabilities $p < o$), we give here a variant of this theorem in which the remaining conditions are also a little alleviated. — The first part of the proof is identic with that of *S. Golab* [5].

Theorem 6. If $f(p, q)$ is an increasing function of q and derivable in (o, o) and $g(p, q)$ has in $q = o$, a derivative with respect to q which is continuous in p where

$$f(p, o) = f(o, p) = p, \tag{13}$$

$$g(M, p) = p, \tag{14}$$

$$g(p, o) = o, \tag{15}$$

then f and g are of the form (1), (2) *with* $\psi(o) = 0$, $\psi(M) = 1$.

Proof: From the derivability conditions supposed and taking also (15) into account we can derive (9) with respect to r in $r = 0$

$$\left[\frac{\partial}{\partial p} f(p, q) = f_1(p, q), \ \frac{\partial}{\partial q} f(p, q) = f_2(p, q), \ \frac{\partial}{\partial q} g(p, q) = g_2(p, q)\right]:$$

$$g_2[f(p, q), o] = f_1(o, o)\, g_2(p, o) + f_2(o, o)\, g_2(q, o).$$

But from (13) and the derivability of f in (o, o)

$$f_1(o, o) = f_2(o, o) = 1$$

follows and thus by defining

$$\psi(p) = g_2(p, o),$$

we have

$$\psi[f(p, q)] = \psi(p) + \psi(q). \tag{16}$$

The function $\psi(p) = g_2(p, o)$ is, with our conditions, continuous. We will show that it is also strictly monotonic and thus prove (1). If there were a pair $p_{\mathrm{I}} < p_{\mathrm{II}}$ such that $\psi(p_{\mathrm{I}}) = \psi(p_{\mathrm{II}})$ then by the continuity of $\psi(p)$ there would be also pairs $p_1 < p_2$ arbitrarily near to each other such that

$$\psi(p_1) = \psi(p_2).$$

But then from (16) also

$$\psi[f(p_1, q)] = \psi[f(p_2, q)].$$

If we let q take all values $q > o$ we see that after $p_2 = f(p_2, o)$ the behaviour of $\psi(p)$ after $p_1 = f(p_1, o)$ must repeat (possibly in a distorted way). As p_1 and p_2 are arbitrarily near to each other this is only possible if $\psi(p)$ is constant and according to (16) this constant must be 0. — But $g_2(p, o) = 0$ is in contradiction with

$$g_2(M, o) = 1,$$

which follows from (14). Thus $\psi(p)$ is (continuous and) strictly monotonic and thus from (16) the assertion (1) follows.

We have also

$$\psi(M) = g_2(M, o) = 1. \tag{17}$$

Substituting (1) into (9) we can proceed as in theorems 4 and 5 :

$$g\{\psi^{-1}[\psi(p) + \psi(q)], r\} = \psi^{-1}\{g[\psi(p, r)], g[\psi(q, r)]\},$$
$$\psi(p) = x, \ \psi(q) = y, \quad \psi\{g[\psi^{-1}(x), r]\} = G(x, r),$$
$$G(x + y, r) = G(x, r) + G(y, r) \text{ (Cauchy [4])}, \ G(x, r) = c(r)\, x,$$
$$g(p, r) = \psi^{-1}[c(r)\, \psi(p)]. \tag{18}$$

If we substitute $p = M$ in this equation, we have from (14) and (17)

$$r = \psi^{-1}[c(r)], \qquad c(r) = \psi(r)$$

and thus (18) becomes

$$g(p, r) = \psi^{-1}[\psi(p)\, \psi(r)], \tag{2}$$

q. e. d.

The derivability conditions present here a rather strong restriction. They exclude e. g. $f(x, y) = \sqrt[k]{x^k + y^k}$, $(k \neq 1)$. Moreover the example $f(x, y) = x + y + \sqrt{xy}$, $g(x, y) = xy$ shows that if instead of derivability only increasing continuity is supposed but (13), (14), 15) are satisfied (e ven from both sides) moreover f and g are commutative and g even of the form (2), f must not be of the form (1). On the other hand this theorem is in a certain sense a complementary of theorems 3 and 5, as here neither commutativity nor associativity are assumed and instead of them the existence of unity- and zero-elements (i. e. the other half of Jánossy's system of axioms) is postulated.

If we suppose also $g(p, M) = p$. then

$$g\,[f,\,(p,\,q),\,r] = h\,[g\,(p,\,r),\,g\,(g,\,r)]$$

can be reduced immediately to (9) by putting $r = M$.

The probability interpretation of this result states that the probabilities $f(p, q)$ resp. $g(p, q)$ of the occurrence of two exclusive events resp. the simultaneous occurrence of two events can be expressed by the formulas (1), (2) from the probabilities q and p of the original exclusive events resp. of the probability q of the first event and the probability p of the second under the supposition that the first has happened, — if the probability-function $f(p, q)$ is increasing with q and derivable in (o, o) (o is the probability of the impossible event) and $g(p, q)$ has in $q = o$ a derivative with respect to q which is continuous in p and if the probability of the impossible event *or* an other event A resp. of the certain event *and* A (without regard to their order), is equal to the probability of A, further if the simultaneous occurrence of any event with the impossible event is also impossible and if finally the probability of (A *or* B) *and* C equals to that of (A *and* C) *or* (B *and* C).

We see that here the independence of the two events which occur simultaneously must not be supposed, because neither commutativity, nor distributivity from *both* sides was supposed. — Thus besides

$$\psi\,[\text{prob}\,(A \text{ or } B)] = \psi\,[\text{prob } A] + \psi\,[\text{prob } B]$$

also

$$\psi\,[\text{prob}\,(A \text{ and } B)] = \psi\,[\text{prob } B] \cdot \psi\,[(\text{prob } A)_{\text{under the condition } B}]$$

holds. Theorem 5 shows that analogously the same result follows if instead of the derivability-conditions and the statements on impossible events above, g is supposed to be non-negative and f continuous increasing and associative (the probability of the occurrence of one of three events does not depend on their arrangement).

Note added in proof: *A. Rényi* has kindly called my attention to a remark in a work of *I. J. Good* (Probability and the weighing of evidence, London 1950, Appendix III., pp. 105—106.), which shows that the author was acquainted with the theorems 2,3 of the present paper. I wish to seize the opportunity to point out that in contrast to the statement of *I. J. Good* my paper[2] does not »rediscover» a result of *N. H. Abel*. The difference is that *Abel* has supposed symmetry and derivability while in the paper[2] these assumptions are not made, instead of them we suppose only continuity. — We make here another remark too: Our considerations at the end of § 6. can be applied also to the system of conditional probabilities introduced by *A. Rényi* (Valószínűségszámítás, Budapest, 1954., 3. függelék, 667—672.). This causes no new difficulties, so we leave the details to the reader.

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РЕШЕНИЕ НЕКОТОРЫХ ПРОБЛЕМ К. БОРШУКА и Л. ЯНОШИ

Я. АЦЕЛЬ

Резюме

В работе было исследовано какие непрерывные, строго монотонные, ассоциативные действия существуют, относительно которых сложение (проблема А. Грузевского - К. Боршука), умножение или квазиумножение $\psi^{-1}[\psi(x)\psi(y)]$ являются дистрибутивными или которые дистрибутивны относительно сложения или квазисложения $\psi^{-1}[\psi(x)+\psi(y)]$

Эти действия соответственно:

$$a\log(a^x+a^y),\ \sqrt[k]{x^k+y^k},\ \psi^{-1}\left[\sqrt[k]{\psi(x)^k+\psi(y)^k}\right],\ cxy,\ \psi^{-1}[c\psi(x)\psi y].$$

Отсюда следует, что можно получить правило сложения или умножения вероятностей в форме по Яноши $\psi^{-1}[\psi(x)+\psi(y)]$; $\psi^{-1}[\psi(x)\psi(y)]$ предполагая из его условий лишь некоторые, а именно: сумма вероятностей непрерывно возрастает со слагающими вероятностями, сумма вероятностей трех событий не зависит от их группировки, произведение вероятностей двух событий положительно и не зависит от их последовательности, и, наконец, вероятность (А либо В) и С равняется вероятности (А и С) либо (В и С). С другой стороны с помощью преобразования одной теоремы С. Голаба получим, что эти же формулы справедливы и в том случае, если кроме последнего условия предположим еще следующее:

Функции, полученные умножением или сложением вероятностей, являются дифференцируемыми (в случае умножения непрерывно по второй переменной); сумма вероятностей – возрастающая; далее, что сумма вероятностей некоторого события А и некоторого невозможного события имеет такое же значение, как и произведение события А на некоторое достоверное событие (в любой последовательности), причем они разны вероятности события А; и наконец, совместное осуществление любого события с невозможным событием тоже невозможно. При этой последней теореме в случае произведения вероятностей не обязательно ограничиваться независимыми событиями. Доказываются дальнейшие подобные теоремы.

THE EFFICIENCY OF SELF-QUENCHING G. M.-COUNTERS FOR COSMIC RAY PARTICLES

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(Presented by L. Jánossy. — Received 10. XII. 1954)

The efficiency of G. M.-counters for cosmic ray particles obtained experimentally was previously found to be smaller than the value expected theoretically. It is shown here that this discrepancy is due to the method employed in the determination of the efficiency. Using dead-time shortening circuits and an apparatus which is sensitive only to the penetrating component of cosmic rays, the efficiency of self-quenching G. M.-counters is found to be greater than 99,9% in good agreement with the value expected theoretically.

Introduction

The efficiency of G. M.-counters for cosmic ray particles is of great importance in the investigations of cosmic rays carried out with counters. The efficiency of G. M.-counters commonly used for cosmic ray work is nearly 100% but can never reach this value.

The error caused by the inefficency of G. M.-counters must generally be taken into account in all experiments, although in many cases it can be neglected compared to other experimental errors. Since the efficiencies of the individual counters connected in coincidence are almost independent, the efficiency of the coincidence arrangement is given by the product of the efficiences of all coincidence counters or counter trays. If the multiplicity of the coincidence arrangement is great this error may be very significant.

Knowledge of the counter efficiency and the use of G. M.-counters of great efficiency is especially important in measurements of non-ionizing particles with anticoincidence arrangements, particulary if the intensity of the non-ionizing component is relatively small compared to that of the ionizing component.

The use of G. M.-counters of great efficiency is not only important in view of a decrease in the size of the corrections in coincidence and anticoincidence measurements and a reduction of background effects, but also so as to decrease the systematic fluctuation of the measurements.

The accuracy of cosmic ray measurements is determined by two parameters : The statistical fluctuations of cosmic rays and the systematic fluctuations of the measured values due either to defects of the electronic apparatus or to the variations of the efficiencies of G. M.-counters.

Measurements of very small effects can be carried out only if the systematic variations are smaller by an order of magnitude than the effect to be measured. Variations due to the electronics can be eliminated only by using an apparatus planned and executed with special care. More difficult is the problem concerning the efficiency of G. M.-counters.

The efficiency of G. M.-counters depends on many parameters and its value changes e. g. with the variation of the counter voltage, of the rate of discharges etc. It must be emphasized that in most cases those parameters vary which may cause inefficiency of the counters and thus we have to speak rather about the variations of the inefficiency. If we can strongly increase the efficiency of counters, i. e. strongly reduce the inefficiency, the fluctuations of this relatively small inefficiency will be smaller, and thus the systematic error of the measurements will be smaller and the limit for the smallest measurable variation will also be lowered. For instance, if we can decrease the inefficiency by one order of magnitude we can measure cosmic ray effects smaller also by one order of magnitude than those measured before.

It can be seen clearly from the above considerations that it is important to know exactly the efficiency of G. M.-counters and to develop such counters and circuits that secure great counter efficiencies.

Measurements carried out with great accuracy to determine the efficiency of G. M.-counters [1], [2], [3], [4] gave a value for the efficiency significantly smaller than the value expected theoretically. The aim of our investigations was to solve the problem of this discrepancy and to determine the real efficiency of G. M.-counters.

The causes of the inefficiency of G. M.-counters

Let us consider the causes of the inefficiency of counters. Counters commonly used — filled with rare gases and organic vapours — give a discharge whenever at least one ion pair is formed in their sensitive volume and during their sensitive time [5]. A particle traversing the sensitive volume of a counter does thus not cause a discharge in the following two cases : 1. The particle does not produce any ion pairs in the gas, 2. The particle traverses the counter during the dead time of the counter.

Let us consider the probability for an ionizing cosmic ray particle traversing the counter not to produce any ion pairs. This probability may be calculated from the value of the primary specific ionization.

Let l be the length of the path of the particle in the gas, ϱ the density of the gas and j_p the primary specific ionization. The distribution of the ionizing collisions is a Poisson distribution and therefore the probability that the particle traverses the path of length l in the gas without producing any ion pairs is

$$w = e^{-j_p \varrho l}$$

and thus the probability that the particle produces at least one ion pair is

$$1 - w = 1 - e^{-j_p \varrho l},$$

If the rate of discharge is small enough so that the dead time of the G. M.-counter can be neglected, $1-w$ is just equal to p, the efficiency of the counter

$$p = 1 - w = 1 - e^{-j_p \varrho l}.$$

The different particles have different path lengths in the counter and therefore the probability for them to cause a discharge in the counter varies accordingly. Thus particles traversing the very edge of the counter travel a shorter way in the gas and therefore the probability for them to cause a discharge in the counter will be significantly smaller than for particles traversing the middle part of the counter. We can get the effective efficiency of the counter by computing the integral mean value of the different efficiencies.

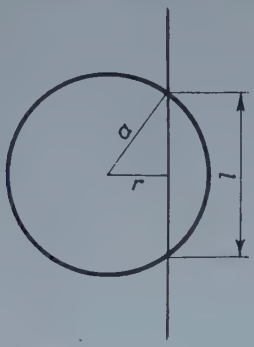

Fig. 1. The length of the path of the particle in the counter

Let a be the radius of the G. M.-counter and r the shortest distance between the path of the traversing particle and the wire of the counter, then the projection of the path in a plane perpendicular to the counter axis has the length (Fig. 1)

$$l = 2\,(a^2 - r^2)^{1/2},$$

which is smaller than or at most equal to the real path length.

Let us consider this projection in first approximation as the length of path in the counter. In this case the probability for the particle to cause a discharge of the counter (i. e. the efficiency of the counter) is according to (1)

$$p_r = 1 - e^{-2 j_p \varrho (a^2 - r^2)^{1/2}}.$$

This expression gives only the lower limit of the real efficiency, because $2\,(a^2-r^2)^{1/2}$, the projection, is also only a lower limit of the real path length.

Integrating this expression from 0 to a and dividing by a, we get the lower limit of the effective efficiency of the counter for isotropic radiation :

$$p_{eff} > 1 - \frac{1}{a}\int_0^a e^{-2j_p\varrho(a^2-r^2)^{1/2}}\,dr. \tag{2}$$

If for reasons of the geometry of the arrangement we need consider only particles passing through the counter inside a certain distance r_1 from the wire, i. e. if we exclude particles passing the edge of the counter, we have to extend the integration only to r_1

$$p_{eff} > 1 - \frac{1}{r_1}\int_0^{r_1} e^{-2j_p\varrho(a^2-r^2)^{1/2}}\,dr\,. \tag{2a}$$

In this case the effective efficiency will be increased in consequence of the exclusion of particles traversing the edge of the counter. Integrals (2) and (2a) may be evaluated approximately by numerical quadrature. Under given geometrical conditions we may have to consider also the anisotropy of the particles.

It can be shown easily by help of expression (2a) that with appropriate choice of the geometrical conditions using G. M.-counters of customary filling and neglecting dead-time effects it is possible theoretically in all cases to get efficiencies greater than 99,9%.

The other reason for counter inefficiency may be that an ionizing cosmic ray particle passes through the counter during dead time. The dead time of counters normally used is of the order of 10^{-4} sec. The probability that a particle should not pass through the counter during the dead time following on the discharge of the counter i. e. the probability that the counter responds to the particle is, as cosmic ray particles are distributed according to a Poisson law,

$$w' = e^{-N\tau},$$

where τ is the dead time of the counter and N the rate of discharge. If $N\tau \ll 1$ which is generally the case in all cosmic ray experiments, then

$$w' = e^{-N\tau} \simeq 1 - N\tau\,.$$

Therefore the efficiency of the counter taking into consideration only the dead-time effect is given by

$$p = 1 - N\tau. \tag{3}$$

The dead-time effect may often be very significant.The discharge rate of counters of 80 cm length used in our experiments was 20/sec and the dead time was $2.\ 10^{-4}$ sec and therefore

$$p = 1 - N\tau = 0{,}996\,,$$

i. e., the efficiency is decreased to 99,6% in consequence of the dead time effect.

The inefficiency due to the dead time may be decreased by using electronic dead time shortening circuits.

The simplest method for shortening the dead time of counters is to use multivibrators that reduce very rapidly the voltage of the counter below the treshold voltage directly the discharge has started and thus prevent the spread of the discharge through the whole tube [6], [7], [8].

If the multivibrator reduces the dead time by one order of magnitude, i. e. if the dead time is $\tau = 2.10^{-5}$ sec, then in the case of counters of 80 cm length the efficiency will be greater than 99,9%, namely

$$p = 1 - N\tau = 0{,}9996\,.$$

To sum up, we can say that by proper choice of the geometrical conditions and by the help of suitable dead time shortening circuits, it is theoretically possible to construct sets of G. M.-counters having efficiencies greater than 99,9%.

Method for measuring the efficiency of counters

An exact method for measuring the efficiency of counters was developed by *Jánossy* and *Rochester* [1] and later by *Jánossy* and *Kiss*. [2, 3]. The principle of the method is as follows (Fig. 2). The counters 1, 2 and 3 are connected in coincidence and the counter X the efficiency of which is to be measured is in

Fig. 2. The principle of the method for measuring the efficiency of counters

anticoincidence. The apparatus measures the number of coincidences (1, 2, 3) and of anticoincidences (1, 2, 3, $-X$) simultaneously.

The coincidences (1, 2, 3) are in general produced by particles traversing all the three coincidence counters and therefore passing also through the counter to be measured. Thus the number of anticoincidences (1, 2, 3, $-X$) immediately shows those events in which the counter to be measured fails to respond; the ratio of the rate of anticoincidences (1, 2, 3, $-X$) and of coincidences (1, 2, 3) represents the inefficiency of the counter. Therefore the efficiency of the counter is given by

$$p = 1 - \frac{(1, 2, 3, -X)}{(1, 2, 3)} . \tag{4}$$

However, coincidences (1, 2, 3) are not only produced by particles traversing the three coincidence counters as well as counter X. There are effects in which particles pass through counters 1, 2 and 3 without traversing counter X. These effects increase the rate of anticoincidences (1, 2, 3, $-X$) and therefore cause the measured value of the efficiency to be smaller than the real one. Among such effects side showers are of the greatest importance (Fig. 3).

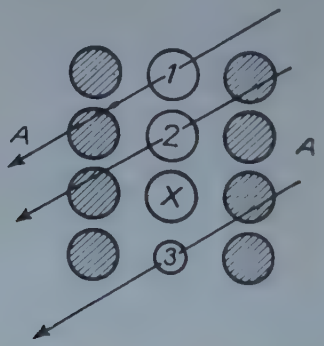

Fig. 3. Anticoincidence counters for eliminating side showers

For the elimination of the disturbing effects due to side showers Jánossy and Rochester, and later Jánossy and Kiss used side counters connected in anticoincidence. Counters A in Fig. 3 are connected in anticoincidence, thus the value of the efficiency in the actual arrangement is given by the following expression

$$p = 1 - \frac{(1, 2, 3, -A, -X)}{(1, 2, 3, -A)} . \tag{5}$$

By the help of this method the efficiency of G. M. counters has been found in most cases to be 99,3%, although in the case of a few very good counters the

efficiency reached a value of 99,6%. The efficiency measured was thus significantly smaller than expected theoretically.

The probable reason for this discrepancy may be the occurrence — apart from side showers — of other processes producing also anticoincidences (1, 2, 3, — *X*). Processes of this kind may be :

1. The scattering of particles,
2. Showers containing electrons as well as photons.

Particles scattered in the frame holding the apparatus or in the counter walls may cause discharges in the coincidence counters 1, 2 and 3 without passing through counter *X* (Fig. 4.). The scattering of penetrating particles, i. e. mesons, is in general negligible [4] but the scattering of soft particles, i. e. electrons may be significant.

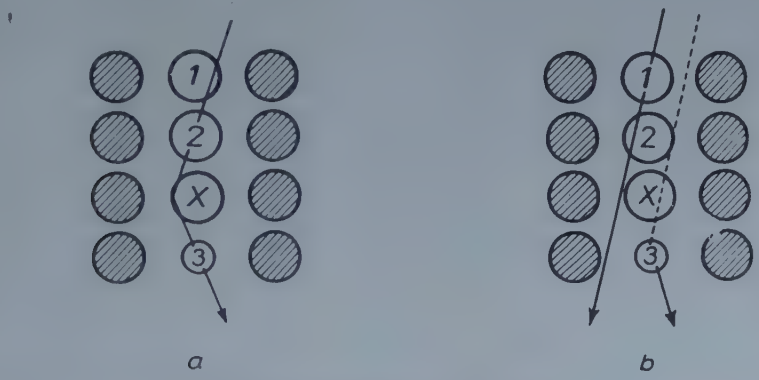

Fig. 4. Effects causing the decrease of the measured value of the counter efficiency : a) The scattering of particles, b) Showers containing electrons as well as photons

Showers coming from above containing electrons as well as photons can produce anticoincidences (1, 2, 3, — *X*) if an electron of the shower causes discharges in counters 1 and 2 and at the same time a photon of the shower produces in the lower wall of the counter to be measured or in the wall of counter 3 an electron which initiates a discharge in counter 3 (Fig. 4/*b*). The probability that a high-energy photon produces an electron in the case of copper counter walls of 1 mm thickness is, roughly estimated, about 10%.

Our interpretation of the discrepancy between the efficiency measured by Jánossy and Kiss and the efficiency expected theoretically is thus that the arrangement employed was not suitable for the exclusion of the above two effects.

Preliminary experiments

The fact that the comparatively great inefficiency found with the arrangement employed by Jánossy and Kiss was not a consequence of the real inefficiency of the counters measured but rather due to the imperfectness of the measu-

ring method, can be proved by using a counter unit of «100%» efficiency in the place of the counter to be measured. This counter unit of «100%» efficiency consists of two G. M.-counters connected in parallel (Fig. 5).

Both counters have an efficiency greater than 99% and therefore the two counters connected in parallel have an efficiency of practically 100%. The efficiency of this unit was found with the arrangement employed by Jánossy and Kiss to be (99,54 $\pm$ 0,03%) i. e. approximately the efficiency found for the best single counters. Our experiment showed beyond doubt the imperfectness of the measuring method.

The elimination of scattered particles and showers containing photons may be attained in a simple way as follows. Since the strongly scattered particles

Fig. 5. Arrangement for measuring the efficiency of two counters connected in parallel

are electrons and the showers containing photons are closely connected with the soft component of cosmic rays containing the electrons, the two disturbing effects may be eliminated in one step by screening out the soft component of cosmic rays. Thus the measurements have to be carried out only with the penetrating component, i. e. with mesons.

The elimination of the soft component may be achived simply by means of Pb absorbers placed between the G. M.-counters. The arrangement used is shown in Fig. 6. This arrangement was at first also checked with a counter unit consisting of two counters connected in parallel.

The efficiency of this unit was found to be (99,97 $\pm$ 0,01%) in good agreement with the value expected. Thus it was shown that this arrangement was suitable for the exact determination of the efficiency of counters.

Experiments and results

We carried out our measurements with the apparatus sketched in Fig. 6. The 10 cm Pb absorber was placed between the counters for the purpose of screening out the soft component. The counters 3 were connected in parallel thus increasing the solid angle of the arrangement.

The counters used in the apparatus and those to be measured had copper cathodes and tungsten wires of 0,1 mm thickness and were filled with argon of 90 mm Hg and alcohol of 10 mm Hg. The diameter of the counters was 4 cm and their length 80 cm.

The anticoincidence arrangement was the same as that used by Jánossy and Kiss [2] the only difference being that the pulse shaping circuit connected to the counter to be measured was a special dead-time shortening circuit (s. c.*). The block diagram of the whole apparatus is shown in Fig. 6.

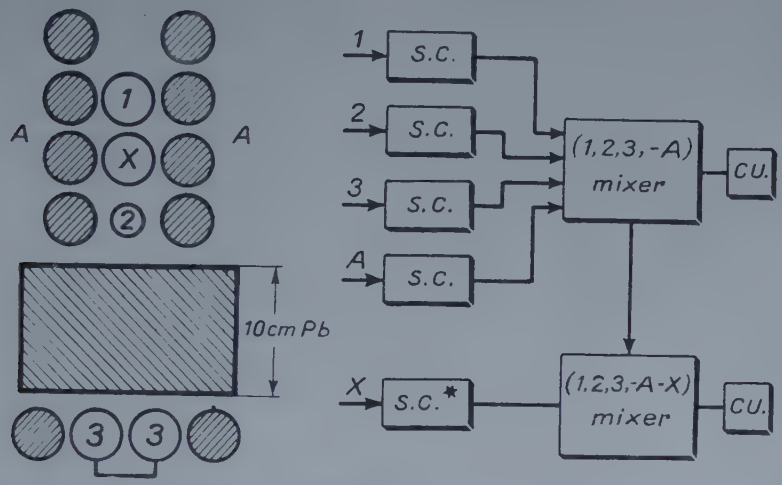

Fig. 6. Arrangement for measuring the efficiency of counters and the block diagram of the anticoincidence circuit. s. c. = shaping circuit, s. c.* = special shaping circuit for dead-time shortening, c. u. = counter unit

The resolving time of the coincidence arrangement was 2 μsec. The dead-time shortening circuit reduced the dead time of the counter to 30 μsec, and its output pulse was 10 μsec wide. During these 10 μ sec-s the anticoincidence circuit was cut off for incoming coincidence impulses. Therefore calculating the dead-time effect we need consider only $\tau = 20\ \mu$ sec, i. e. the part of the dead-time beginning after the output pulse.

In the case of G. M.-counters of 80 cm length and a rate of discharge of 20/sec the efficiency is, taking into account only the dead-time effect, according to (3)

$$p = 0{,}9996\,,$$

i. e. the efficiency is 99,96%.

The counters to be measured were placed between two counters of diameters 4 cm and 3,5 cm resp. With the geometry defined by these two counters the lower limit of the effective efficiency of the counter to be measured, taking into account only the ionization effect, was according to (2a)

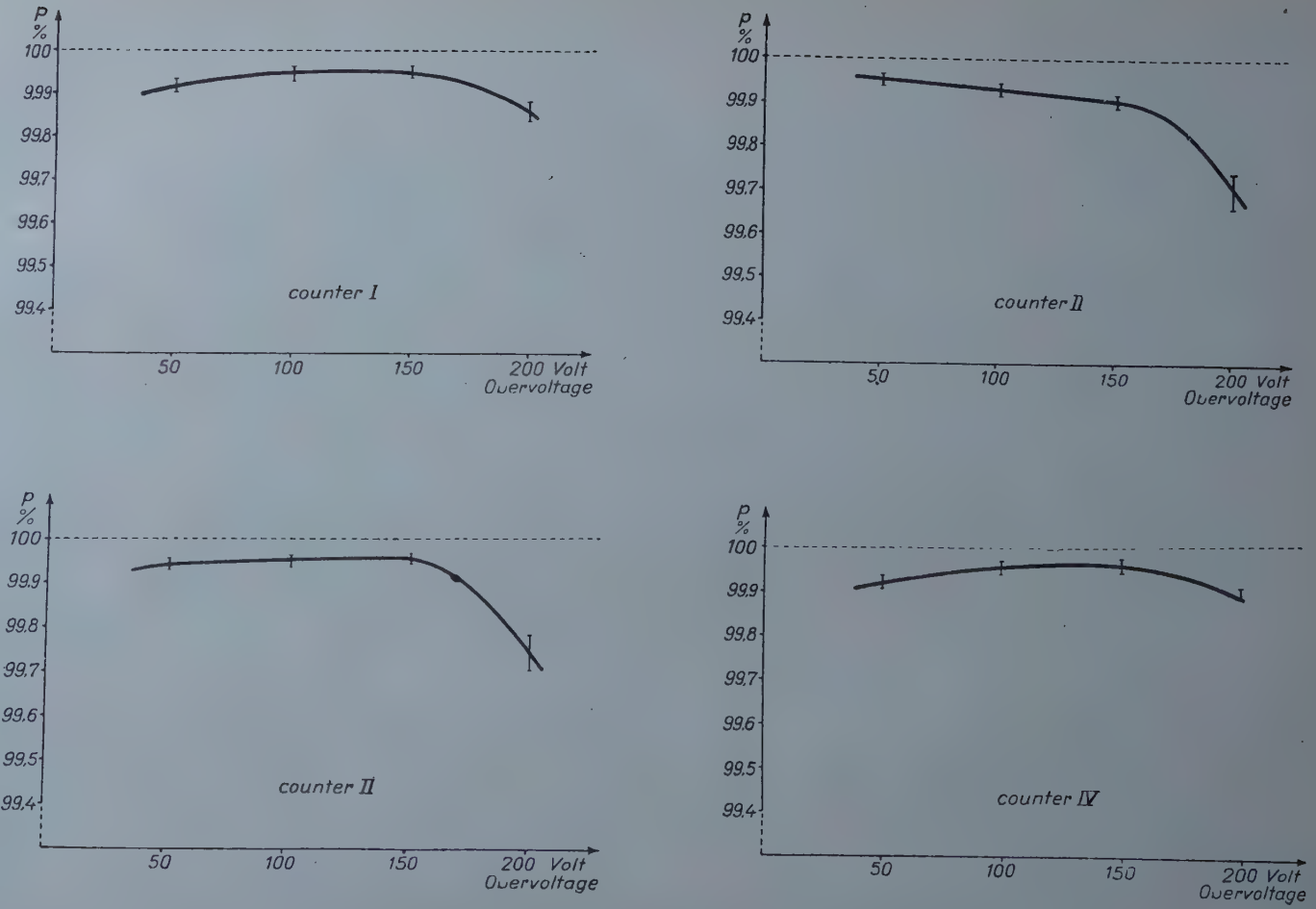

Fig. 7. The efficiencies of four counters as functions of the overvoltages

$$p_{eff} > 0{,}9999\,,$$

i. e. greater than 99,99%.

Thus under the conditions described above we expect theoretically the efficiency of G. M.-counters to be about 99,95%.

We measured the efficiency of 28 G. M.-counters at overvoltages of 50, 100, 150 and 200 V and plotted the efficiency as a function of the overvoltage.

The mean value of the efficiency of 19 G. M.-counters out of the 28 measured was about 99,9% in contrast to the value of 99,3% found by Jánossy and Kiss. These 19 G. M.-counters were qualified as «good» counters and the efficiency of 4 of these «good» counters was determined very exactly. (Table I, Fig. 7).

TABLE I

Overvoltage	50V	100V	150V	200V
Efficiency %				
Counter I	99,92 ± 0,01	99,94 ± 0,01	99,95 ± 0,01	99,86 ± 0,02
Counter II	99,94 ± 0,01	99,94 ± 0,01	99,95 ± 0,01	99,74 ± 0,04
Counter III	99,96 ± 0,01	99,94 ± 0,01	99,92 ± 0,01	99,71 ± 0,04
Counter IV	99,93 ± 0,01	99,95 ± 0,01	99,95 ± 0,01	99,88 ± 0,02

It is clear from the Table and the graphs, that the efficiency of each counter is significantly greater than 99,9% at one or more values of overvoltages, and the maximum values of their efficiencies are close to 99,95%, the value theoretically expected. By the term «significantly greater than 99,9%» we mean that considering even a fluctuation as great as three times the standard error, the efficiency will be greater than 99,9%.

As a result of our measurements we can state that using dead-time shortening circuits the efficiency of self-quenching counters for ionizing cosmic ray particles is greater than 99,9%. This result is in good agreement with the efficiency evaluated theoretically.

I am indebted to Prof. *L. Jánossy* who raised the problem and helped my work with valuable discussions. My thanks are due to *G. Bozóki* who helped to build the apparatus. I am also indebted to him and to *Erika Halmos* for their assistance in the course of the measurements.

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ВЕРОЯТНОСТЬ СРАБАТЫВАНИЯ САМОГАСАЮЩИХСЯ СЧЕТЧИКОВ Г. - М. ДЛЯ ЧАСТИЦ КОСМИЧЕСКОГО ИЗЛУЧЕНИЯ

Э. ФЕНЬВЕШ

Р е з ю м е

Величина вероятности срабатывания самогасающихся счетчиков Гейгера Моллера для частиц космического излучения на основе произведенных до сих пор измерений оказалась меньшей, чем ожидаемое значение, вытекающее из теоретических рассуждений. Можно показать, что это расхождение является следствие м примененного метода измерений. Если при измерении применять схему, укорачивающую мертвое время счетчиков Г.—М., и вести измерение вероятности срабатывания только проходящим компонентом космического излучения, то вероятность срабатывания самогасающихся счетчиков Гейгера – Моллера будет больше, чем 99,9%, что хорошо совпадает с теоретически ожидаемым значением.

КРАТКИЕ СООБЩЕНИЯ
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ON SOME PROBLEMS OF THE OPERATION OF MICROTRONS

By

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A microtron operates in the following manner. A microwave resonant cavity is placed near the edge of a steady, uniform magnetic field, with its axis of symmetry perpendicular to the direction of the magnetic field. The resonator is excited so that the peak voltage across the lips is slightly larger than the voltage corresponding to the rest mass of the electron (about 0,511 MeV). Electrons are emitted from one of the lips by field emission and those which cross the gap at the appropriate phase of the electric field emerge from the cavity with a total energy of two rest masses. The values of the magnetic field and the operating frequency of the cavity are adjusted so that such electrons require a time corresponding to two cycles of the radio-frequency field to complete their first orbit. They will then make the next transit of the cavity at the correct phase for each electron to gain one additional rest mass of energy. Since the time needed for an electron to complete an orbit is proportional to its total energy, the corresponding time interval is now three cycles of the radiofrequency field. Thus the electrons under consideration once more arrive at the cavity at the appropriate phase for each to receive again one rest mass of energy ; etc.

The most important characteristics of an accelerator are the final energy, the energy spread and the intensity of the accelerated beam. The final energy in a microtron [1], [2], can be easily calculated in terms of the diameter of the magnet by using the resonance conditions for accelerating voltage and magnetic field :

$$V_r = \frac{m_0 c^2 + E_0}{e} \frac{n}{l - n},$$

$$B_r = \frac{2\pi (m_0 c^2 + E_0)}{ec} \frac{1}{l - n},$$

where $l > n$, l and n are arbitrary integers, E_0 is the kinetic energy of the electron at start, B_r the magnetic flux density, and λ the wave-length.

The energy spread and the current intensity can be obtained only by a detailed examination of the motion of electrons. Such studies have been carried

out by *Schmelzer* [3] and *Henderson* et al. [4], who determined by a graphical method a phase region with the following features : electrons starting within this phase interval are being accelerated each time they traverse the r.-f. field and never encounter a decelerating field so that they can travel indefinitely in their orbit. The phase region thus determined is narrower than necessary from a practical point of view. On the one hand electrons reach their final energy and leave the accelerator after a relatively small number of turns and on the other hand it is also possible that electrons become well phased by losing energy on one occasion and that they will thus be accelerated to the desired energy. Therefore the only possible way to approach the problem of energy spread is to follow the motion of single electrons throughout their orbits and determine at each turn their entering phase and energy. This can be done by a simple graphical method similar in some respects to that due to *Slater* and *Terrall* [5], for linear accelerators.

If an electron enters the r.-f. gap for the k-th acceleration at the phase φ_k with an energy ε_k (measured in units of the rest energy m_0c^2), then the next entrance will occur at the phase :

$$\varphi_{k+1} = \varphi_k + 2\pi \frac{V_0}{V_r} \sin \varphi_k + 2\pi \frac{m_0 c^2}{eV_r} \varepsilon_k ,$$

with the energy :

$$\varepsilon_{k+1} = \varepsilon_k + \frac{eV_0}{m_0 c^2} \sin \varphi_k .$$

V_r is the «resonance voltage» satisfying the resonance conditions, V_0 the amplitude of the accelerating voltage.

Plotting the curves $\varphi_{k+1} = C_1 = \text{constant}$ and $\varepsilon_{k+1} = C_2 = \text{constant}$ in the $(\varphi_k, \varepsilon_k)$ coordinate system we can read off the entering phases and energies of the electron. Since the curves corresponding to various values of the constants C_1, C_2 can be obtained by shifting the curve along the ε-axis, it is sufficient to plot one φ_{k+1} — and one ε_{k+1} — curve, the next values of ε and φ can be determined step by step by proper translation of the curves ε_{k+1} and φ_{k+1}. (The constants C_1 and C_2 are given by the point of intersection of the respective curves with the ε axis.)

In this manner taking for example a 4 MeV accelerator ($V_r = 0{,}5$ MV, $B_r = 1070$ gauss, magnet diameter 30 cm) and choosing the optimum value $V_r/V_0 = 0{,}98$, we obtained as probable energy spread 1,2% and mean electron current intensity 10 μA.

Although no details are available concerning the operational conditions of microtrons actually in use, our results seem to be in good agreement with the experimental data we found in the literature. For example *Henderson* [6],

obtained 4,5 + 0,07 MeV (1,5% energy spread) and *Redhead* [2], obtained ½ μA electron current.

The discrepancy in the beam current intensity can be accounted for by defocusing effects of various origin.

The above considerations show that the energy homogeneity of the beam depends on the operational conditions of the microtron. It can be shown that the microtron is capable of yielding much narrower energy sprectra than linear accelerators of the same final energy, which is a further advantage of microtrons.

The author wishes to thank Dr. *P. Faragó* for his valuable advice.

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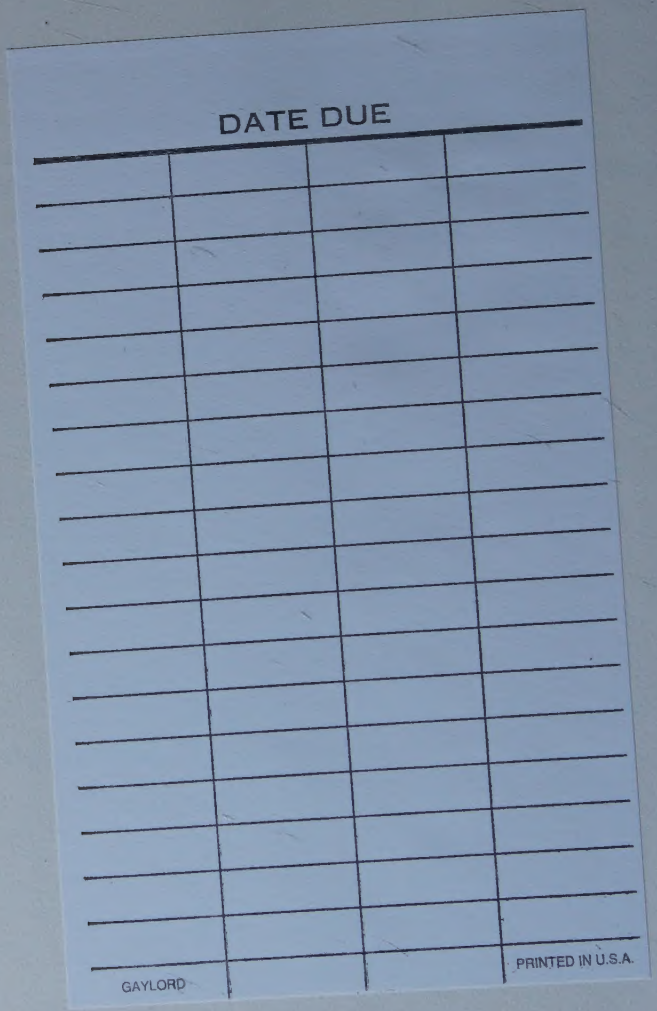
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